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PURDUE UNIVERSITY

DEPARTMENT OF STATISTICS

DIVISION OF MATHEMATICAL SCIENCES

ON SOME MULTIPLE DECISION PROBLEMS

by

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INTRODUCTION

In practice, there often do arise situations in which the researcher wishes to choose the best (or the t best) from a group of k populations, where the "bestness" of a population is based on ranking them according to the value of some characteristic of interest. In such situations, the classical tests of homogeneity are inadequate in the sense that they have not been designed to answer several possible questions in which the researcher may really be interested. Bahadur [1], Mosteller [54] and Paulson [59] were among the first research workers to recognize the inadequacy of the usual tests of homogeneity and to consider more meaningful formulations in order to answer these questions. These developments set the stage for the early investigations of multiple decision problem which have now come to be known as selection and ranking problems.

In the theory of selection and ranking procedures, there are two basic formulations to the problem. The first one is called the indifference zone formulation due to Bechhofer [13] and the other is the subset formulation due to Gupta [29]. The goal of the basic problem in the formulation of Bechhofer is to choose one of the populations as the best. The researcher is required to specify an "indifference zone" in the parameter space and the

procedure determines the smallest sample size so that a certain
probability condition is satisfied whenever the unknown parameters
lie in the "preference zone". For example, if we are interested
in selecting the population with the largest mean in $N(\theta_i,1)$,
$i=1,\ldots,k$, the indifference zone is the set of all θ's such that
the largest and the second largest differ by an amount $\leq \delta^*$,
whereas the "preference zone" is the set of all θ's such that the
above difference $> \delta^*$. Other contributions to this indifference
zone formulation have been made by Bechhofer and Sobel [16],
Sobel and Huyett [70], Barr and Rizvi [12], Desu and Sobel [22],
Bechhofer, Kiefer and Sobel [15], Santner [64] and others. The
goal of the basic problem in the subset selection formulation of
Gupta is to select a subset of the given populations which depends
on the outcome of the experiments and is not fixed in advance
such that it includes the best population with a specified minimum
probability regardless of the unknown configuration of parameters,
i.e., over the whole parameter space. Some recent results in the
area of subset selection formulation are Gnanadesikan and Gupta [28],
Gupta and Studden [43], Gupta and Panchapakesan [38],
Gupta and Santner [41], Huang [46] and Wong [72].

Many problems in reliability can be considered in the context
of selection and ranking problems. For example, one may wish to
choose one or more of the several systems or components which has
the largest mean life or the largest median life. In general,
reliability problems also deal where the distributions are unknown
but assumed to belong to a class of distributions such as that

having an increasing failure rate (IFR). Such distributions form
a special cases of what are now commonly known as restricted
families of probability distributions. The investigations of
Barlow and Gupta [5] form the initial efforts on ranking problems
for such families. Other contributions to selection problem for
the restricted families of probability distributions are Gupta and
Panchapakesan [39,40] and Patel [58]. This area of research still
remains largely unexplored.

The main investigation of this thesis is to propose and study
selection procedures for some problems.

Chapter I deals with some selection and ranking procedures
for restricted families of probability distributions. In
Section 1.1, definitions of various partial orderings on the
space of distributions are given. In Section 1.2, we propose and
study a selection rule for distributions which are convex-ordered
with respect to specified distribution G. Some properties of this
selection rule are discussed. The asymptotic relative efficiencies
of this rule with respect to some other selection rules are derived.
Section 1.3 deals with the selecting the best population using the
indifference zone approach. In Section 1.5, we propose and study
a selection rule for distributions which are s-ordered with
respect to G where we are interested in the scale parameter case.
The estimation of ordered parameters from the k unknown distribu-
tions is discussed in Section 1.8.

Chapter II discusses some interval estimation problems from
k populations. We are interested in finding the smallest sample

size N to be chosen from each population such that the probability
that a given confidence interval I which is based on $T_{[k]} = \max\limits_{1 \le i \le k} T_i$
(T_i is an appropriate statistic from i-th population) contains at
least one good population, is at least P*, where P* is a specified
number, $0 < P* < 1$. Also we are interested in finding the smallest
sample size N such that the probability that I contains all good
populations (or excludes all bad populations) is at least P*.
Section 2.2 deals with the above problems for the location parameter
case. The infima of coverage probabilities are obtained. For scale
parameter case, the above problems are investigated in Section 2.3.
In Section 2.4, we illustrate the above results by means of two
examples.

In some situations, one has to deal with a vector-valued
parameter $\underline{\lambda}_i$ associated $\underline{X}_i$. In such cases one may consider
comparing populations or $\underline{\lambda}_i$'s in terms of majorization and weak
majorization of these vectors. Chapter III deals with such
problems. The parameter space is partially ordered by means of
majorization or weak majorization. Selection procedures are
proposed and studied. In Section 3.2, a class of procedures
R_h for selecting the best population is defined. A sufficient
condition is obtained for the infimum of the probability of a
correct selection to be Schur-convex in $\underline{\lambda}$. Also another sufficient
condition for the same infimum of the probability of a correct
selection to be nondecreasing and Schur-convex in $\underline{\lambda}$ is obtained
in Section 3.3. Section 3.5 and 3.6 deal with selection procedures

for multivariate normal distributions in terms of majorization
and weak majorization. Various cases corresponding to the
known or unknown common covariance matrix Σ are studied. Properties
of these selection procedures are also established.

CHAPTER I

SELECTION PROCEDURES FOR RESTRICTED FAMILIES
OF PROBABILITY DISTRIBUTIONS

1.1 Introduction

In many problems, especially those in reliability theory, one is interested in using a model for life length distribution which belongs, for example, to a family of distributions having increasing failure rate (IFR), or increasing failure rate on the average (IFRA). Such distributions form special cases of what are now commonly known as restricted families of probability distributions. These are defined more precisely later in this section. The idea of using such families stems from the fact that in many cases the experimenter cannot specify the model (distribution) exactly but is able to say whether it comes from a family of distributions such as IFR, IFRA. Families of probability distributions of these types have been studied by several authors, see, for example, Barlow, Marshall and Proschan [7]. These authors have mainly concerned themselves with probabilistic aspects of these distributions. To some extent, there have been some investigations dealing with statistical inference for some of these families; see for example, Barlow and Proschan [8], [9], Barlow and Doksum [3].

In this chapter we are interested in studying multiple decision procedures for k (k $\geq$ 2) populations which are themselves unknown but which are assumed to belong to a restricted family. First we give some notations and definitions. A binary ordering relation ($\prec$) is called a partial ordering in the space of probability distributions if

(a)   F $\prec$ F for all distributions F, and

(b)   F $\prec$ G, G $\prec$ H imply F $\prec$ H.

Note that F $\prec$ G and G $\prec$ F do not necessarily imply F = G.
We now define some of the special order relations of interest to us (see Barlow and Gupta [5]).

(i)   F is said to be convex with respect to G (written F $\underset{c}{\prec}$ G) if and only if $G^{-1}F(x)$ is convex on the support of F.

(ii)   F is said to be star-shaped with respect to G (written F $\underset{*}{\prec}$ G) if and only if F(0) = G(0) = 0 and $\dfrac{G^{-1}F(x)}{x}$ is increasing in x $\geq$ 0 on the support of F.

(iii)   F is said to be r-ordered with respect to G (F $\underset{r}{\prec}$ G) if and only if F(0) = G(0) = $\dfrac{1}{2}$ and $\dfrac{G^{-1}F(x)}{x}$ is increasing (decreasing) for x positive (negative) on the support of F.

(iv)   F is said to be s-ordered with respect to G (F $\underset{s}{\prec}$ G) if and only if F(0) = G(0) = $\dfrac{1}{2}$ and $G^{-1}F$ is concave-convex about the origin, on the support of F.

If $G(x) = 1-e^{-\lambda x}$, x $\geq$ 0, then F $\underset{c}{\prec}$ G is equivalent to saying that F has increasing failure rate (IFR). The class of IFR distributions has been studied by Barlow, Marshall and Proschan [7].

Again if $G(x) = 1-e^{-\lambda x}$, $x \geq 0$, $F \underset{*}{\prec} G$ is equivalent to saying that F has increasing failure rate on average (IFRA). The class of IFRA distributions has been studied by Barlow, Esary and Marshall [4]. The r-ordering has been defined and investigated by Lawrence [48]. The s-ordering and c-ordering have been studied by Van Zwet [71] and Lawrence [48].

In the statistical literature, selection problems for restricted families were first investigated by Barlow and Gupta [5]. Some further results in this direction and a review of some important results concerning inequalities for restricted families and problems of inference for such families have been given by Gupta and Panchapakesan [39]. The selection of the population with largest α-quantile from distributions which are star-shaped with respect to the folded normal distribution has been considered by Gupta and Panchapakesan [40]. In a recent paper, Patel [58] has studied the selection of IFR populations which differ only in the scale parameters. His procedure is based on the total life statistic until r-th failure.

In Section 1.2, we propose and study a selection rule for distributions which are $\prec_c$ ordered with respect to a specified distribution G assuming there exists a best one. Some properties of this rule are discussed. The infimum of the probability of a correct selection is obtained and an asymptotic expression is also given. We also study the asymptotic relative efficiencies of this rule with respect to some selection procedures. Section 1.3

deals with the selecting the best population in the frame work of Section 1.2 using the indifference zone approach. In Section 1.4, we propose a selection procedure for distributions that are $\prec_*$ ordered with respect to G. Section 1.5 deals with selection procedure with respect to the means for distributions that are s-ordered with respect to G where we are interested in the scale parameter case. The distribution of V_i (see Section 1.5) is also investigated. Section 1.8 deals with estimation of ordered parameters from k unknown distributions where we are interested in the scale parameter case.

1.2 Selection rules for distributions $\prec_c$ ordered with respect to a specified distribution G.

Let $\mathfrak{J}$ be the class of absolutely continuous distribution functions F on R with positive and right-(or left-) continuous density f on the interval where $0 < F < 1$. It follows that the inverse function F^{-1} is uniquely determined on (0,1). We take $F^{-1}(0)$ and $F^{-1}(1)$ to be equal to the left hand and right hand endpoints of the support of F. For F, G $\in \mathfrak{J}$, consider the following transformation (see Barlow and Doksum [3])

$$(1.2.1) \qquad H_F^{-1}(t) = \int_{F^{-1}(0)}^{F^{-1}(t)} g[G^{-1}F(u)]du, \qquad 0 \leq t \leq 1.$$

We assume that G is always fixed. Since H_F^{-1} (the inverse of H) is strictly increasing on [0,1], H_F is a distribution. We know that $F \prec_c G$ if and only if H_F is convex on the interval where $0 < H_F < 1$.

Since G is assumed known we can estimate H_F^{-1} by substituting the empirical distribution F_n of F; that is

$$(1.2.2) \qquad H_n^{-1}(t) = H_{F_n}^{-1}(t) = \int_{F^{-1}(0)}^{F_n^{-1}(t)} g[G^{-1}F_n(u)]du$$

and

$$(1.2.3) \qquad H_n^{-1}(\frac{r}{n}) = \int_{F^{-1}(0)}^{X_{r,n}} g[G^{-1}F_n(u)]du = \sum_{i=1}^{r} g[G^{-1}(\frac{i-1}{n})](X_{i,n}-X_{i-1,n})$$

where $X_{i,n}$ is the i-th order statistic in a sample of size n from F and $X_{0,n} \equiv 0$.

If $G(x) = 1-e^{-x}$ for $x \geq 0$, then (1.2.3) can be written as

$$(1.2.4) \qquad H_n^{-1}(\frac{r}{n}) = \frac{1}{n}[X_{1,n} +...+ X_{r-1,n} + (n-r+1)X_{r,n}].$$

We say that $X_{1,n} +...+ X_{r-1,n} + (n-r+1)X_{r,n}$ is the total life statistic until r-th failure from F. Thus, $H_n^{-1}(\frac{r}{n}) = \frac{1}{n} \{ \begin{smallmatrix} \text{total life statistic} \\ \text{until r-th failure} \end{smallmatrix} \}$.

(A) Selection procedure and its properties

Let $\pi_1,...,\pi_k$ be k populations. The random variable X_i associated with π_i has distribution function F_i, $i = 1,2,...,k$, where $F_i \in \mathcal{F}$ (i = 1,...,k). Let $F_{[k]}$ denote the cumulative distribution function (c.d.f.) of the "best" population. For example, if we are interested in the quantile selection problem, the "best" population may be defined as that $F_{[k]}$ (unknown) which satisfies (a) below. We assume that

(a) $F_{[i]}(x) \cdot F_{[k]}(x)$ for all x, $i = 1,...,k-1$;

(b)   there exists a distribution G such that

$$F_{[i]} \precsim G \qquad i = 1,\ldots,k,$$

where $\prec$ denotes a partial ordering relation in the space of probability distributions. It should be pointed out that the condition (a) above may also imply that $F_{[k]}$ is the distribution with the largest (smallest) parameter. For example, if

$F_i(x) = F(\frac{x}{\theta_i})$ for $x \geq 0$, $\theta_i > 0$ $(i = 1,\ldots,k)$, then $F_{[k]}$ is the distribution with the largest θ_i. We are given a sample of size n from each π_i $(i = 1,\ldots,k)$. Our goal is to select a subset from the k populations so as to include the population with $F_{[k]}$. Let $\mathcal{L} = \{F = (F_1,\ldots,F_k) : \exists \ a j \ \text{such that} \ F_i(x) \geq F_j(x) \ \forall \ i \neq j\}$. Let

$$(1.2.5) \qquad T_i = \sum_{j=1}^{r} a_j \, X_{i;j,n} \qquad \text{for } i = 1,\ldots,k.$$

$$(1.2.6) \qquad T = \sum_{j=1}^{r} a_j \, Y_{j,n}$$

where $X_{i;j,n}$ is the j-th order statistic from F_i, $Y_{j,n}$ is the j-th order statistic from G, r is a fixed positive integer $(1 \leq r \leq n)$,

$$a_j = g \, G^{-1}(\frac{j-1}{n}) - g \, G^{-1}(\frac{j}{n}) \qquad \text{for } j = 1,\ldots,r-1$$

and $a_r = g \, G^{-1}(\frac{r-1}{n})$.

For selecting a subset containing $F_{[k]}$, we propose the selection rule R_1 as follows:

R_1:   Select population π_i if and only if

$$(1.2.7) \qquad T_i \leq c_1 \max_{1 \leq j \leq k} T_j$$

where $c_1 = c_1(k,P^*,n,r)$ is some number between 0 and 1 which is determined as to satisfy the probability requirement

$$(1.2.8) \qquad \inf P\{CS|R_1\} \geq P^*$$

where CS stands for a correct selection, i.e., the selection of any subset which contains the population with distribution $F_{[k]}$. Let $T_{(i)}$ be associated with $F_{[i]}$ and $W_i(x)$ be the c.d.f. of $T_{(i)}$ from $F_{[i]}$. We now state a few preliminary lemmas.

Lemma 1.2.1. (Lehman [49] p. 112) Let $F(x)$ be a distribution function on the real line. If $\psi(x)$ is any nondecreasing function of x, then $E \psi(X)$ is a nonincreasing function of F, i.e., if $F_1(x) \leq F_2(x)$ for all x, then $\int \psi(x)dF_2(x) \leq \int \psi(x)dF_1(x)$.

Lemma 1.2.2. Let $X_1,\ldots,X_n$ be i.i.d. with distribution $F(x)$. Let ψ be a function of $X_1,\ldots,X_n$ which is nondecreasing in each of its arguments. Then $E\psi(X_1,\ldots,X_n)$ is nonincreasing function of F.

Lemma 1.2.3. (Gupta and McDonald [35]) Let $\underline{X} = (X_{11},\ldots,X_{1n_1},\ldots, X_{k1},\ldots,X_{kn_k})$ be a vector valued random variable of $\sum_{i=1}^{k} n_i \ (\geq 1)$ independent components with X_{ij} having the distribution $F_i(x)$, $j = 1,\ldots,n_i$, $i = 1,\ldots,k$. Let ψ be a function of $X_{11},\ldots,X_{1n_1},\ldots, X_{k1},\ldots,X_{kn_k}$ which, for any fixed i, is a nondecreasing (nonincreasing) function of $X_{i1},\ldots,X_{in_i}$ when the other components of $\underline{X}$ are held fixed. Then $E\psi(\underline{X})$ is a nonincreasing (nondecreasing) function of F_i.

The following two lemmas are from Barlow and Proschan [8].

Lemma 1.2.4. (a) $\phi(\sum_{i=1}^{n} a_i X_i) - \phi(0) \leq \sum_{i=1}^{n} a_i [\phi(X_i) - \phi(0)]$ for all

$0 \leq X_1 \leq \ldots \leq X_n \leq b$ and for all convex ϕ on $[0,b]$ if and only if

$0 \leq \bar{A}_i = \sum_{j=i}^{n} a_j \leq 1$ for $i = 1, 2, \ldots, n$.

(b) $\phi(\sum_{i=1}^{n} a_i X_i) - \phi(0) \geq \sum_{i=1}^{n} a_i [\phi(X_i) - \phi(0)]$ for all $0 \leq X_1 \leq \ldots \leq X_n$

and for all convex ϕ on $(-\infty, \infty)$ if and only if $\bar{A}_1 \geq 1$, $\bar{A}_2 \geq 1, \ldots,$

$\bar{A}_k \geq 1$, $\bar{A}_{k+1} \leq 0, \ldots, \bar{A}_n \leq 0$ for some k $(0 \leq k \leq n)$.

Lemma 1.2.5. (a) $\phi(\sum_{i=1}^{n} a_i X_i) \leq \sum_{i=1}^{n} a_i \phi(X_i)$

for all star-shaped ϕ on $[0,b]$ and all $0 \leq X_1 \leq \ldots \leq X_n \leq b$ if and

only if there exists k $(1 \leq k \leq n)$ such that

$0 \leq \bar{A}_1 \leq \ldots \leq \bar{A}_k \leq 1$ and $\bar{A}_{k+1} = \ldots = \bar{A}_n = 0$.

(b) $\phi(\sum_{i=1}^{n} a_i X_i) \geq \sum_{i=1}^{n} a_i \phi(X_i)$ for all $0 \leq X_1 \leq \ldots \leq X_n$ and for all

star-shaped ϕ on $(-\infty, \infty)$ if and only if there exists k $(1 \leq k \leq n)$

such that

$$\bar{A}_1 \geq \ldots \geq \bar{A}_k \geq 1; \quad \bar{A}_{k+1} = \ldots = \bar{A}_n = 0.$$

Lemma 1.2.6. Let F_1, F_2 be two distribution functions such that

$F_1(x) \geq F_2(x) \ \forall x$.

Let $T_i = \sum_{j \in \Delta} b_j X_{i;j,n}$ $\quad i = 1, 2$

where $b_j > 0$ for $j \in \Delta$, $\Delta \subset \{1, 2, \ldots, n\}$ and $X_{i;j,n}$ is the j-th

order statistic from F_i, $i = 1, 2$.

Then $\quad P[T_1 \leq x] \geq P[T_2 \leq x]$.

Proof.

$$\text{Let } \psi_i\ (X_{i1},\ldots,X_{in}) = \begin{cases} 1 & \text{if } T_i \geq x \\ \\ 0 & \text{otherwise} \end{cases}$$

where $X_{i1},\ldots,X_{in}$ are n observations from F_i (i = 1,2).

Since $\psi_i\ (X_{i1},\ldots,X_{in})$ is nondecreasing in each of its

arguments, by Lemma 1.2.2 we have

$$E\ \psi_1(X_{11},\ldots,X_{1n}) \leq E\ \psi_2(X_{21},\ldots,X_{2n})$$

That is $\quad P[T_1 \geq x] \leq P[T_2 \geq x]$

This proves the Lemma.

We now state and prove following theorem which is more general than

that of Patel [58].

Theorem 1.2.1. Let F_i, $G \in \mathfrak{s}$, $F_i(x) \geq F_{[k]}(x)$ $\forall$ x and i = 1,2,...,k,

$F_{[k]}(0) = 0$ and $F_{[k]} \underset{c}{\prec} G$.

If $a_j \geq 0$ for j = 1,2,...,r, $G^{-1}(0) \leq 0$, g $G^{-1}(0) \leq 1$ and $a_r \geq c_1$,

then

$$(1.2.9) \qquad P[CS|R_1] \geq \int_{G^{-1}(0)}^{\infty} G_T^{k-1}(\frac{x}{c_1})dG_T(x)$$

where $G_T(x)$ is the c.d.f. of T.

Proof. $\quad P[CS|R_1] = P[T_{(k)} \geq c_1\ T_{(i)},\ i \neq k]$

$$= \int_0^{\infty} \prod_{i=1}^{k-1} W_i^{k-1}(\frac{x}{c_1})dW_k(x)$$

$$\geq \int_0^{\infty} W_k^k(\frac{x}{c_1})dW_k(x) \quad \text{(By Lemma 1.2.6)}$$

$$= P[Z_k \geq c_1Z_j,\ j \neq k]$$

where $Z_1,\ldots,Z_k$ are i.i.d. with c.d.f. $W_k(x)$.

Let $\varphi(x) = G^{-1}F_{[k]}(x)$.

Note that

$$(1.2.10) \qquad Z_i \underset{st}{} \sum_{j=1}^{r} a_j X^*_{i;j,n} \qquad i = 1,\ldots,k,$$

where $X^*_{i;j,n}$ is the j-th order statistic in a sample of size n from $F_{[k]}$, $i = 1,\ldots,k$.

$$(1.2.11) \qquad P[Z_k \geq c_1 \max_{1 \leq j \leq k} Z_j] = P[\varphi(\tfrac{1}{c_1} Z_k) \geq \varphi(Z_i), \ i = 1,\ldots,k-1]$$

Since $\sum_{j=1}^{r} a_j = g\, G^{-1}(0) \leq 1$ and $a_j \geq 0 \ \forall j = 1,\ldots,r$, then, by Lemma 1.2.4 (a) and (1.2.10),

$$(1.2.12) \qquad \varphi(Z_i) \leq \sum_{j=1}^{r} a_j\, \varphi(X^*_{i;j,n}).$$

Since $\tfrac{1}{c_1} a_r \geq 1$ and $\tfrac{1}{c_1} \sum_{j=i}^{r} a_j \geq 1$ for $i = 1,\ldots,r$, we have, by Lemma 1.2.4 (b) and (1.2.10),

$$(1.2.13) \qquad \varphi(\tfrac{1}{c_1} Z_k) \geq \tfrac{1}{c_1} \sum_{j=1}^{r} a_j\, \varphi(X^*_{k;j,n}).$$

$$(1.2.14) \qquad \varphi(X^*_{i;j,n}) \underset{st}{=} Y_{i;j,n}$$

where $Y_{i;j,n}$ is the j-th order statistic from G, $i = 1,2,\ldots,k$.

Thus from (1.2.11), (1.2.12), (1.2.13) and (1.2.14),

$$P[Z_k \geq c_1 \max_{1 \leq i \leq k} Z_i] \geq P[\sum_{j=1}^{r} a_j Y_{k;j,n} \geq c_1 \sum_{j=1}^{r} a_j Y_{i;j,n},$$
$$i = 1,2,\ldots,k-1]$$

$$= \int_{G^{-1}(0)}^{\infty} G_T^{k-1}(\tfrac{x}{c_1}) dG_T(x)$$

This completes the proof.

The constant $c_1 = c_1(k,P^*,n,r)$ satisfying (1.2.8) is determined by

$$\int_{G^{-1}(0)}^{\infty} G_T^{k-1}(\frac{x}{c_1})dG_T(x) \geq P^* \quad \text{and}$$

$$g\ G^{-1}(\frac{r-1}{n}) \geq c_1.$$

We now consider two specific distributions $G(x)$. If $G(x) = 1-e^{-x}$, $x \geq 0$, then we have following result which generalizes the result of Patel [58].

<u>Corollary 1.2.1.</u>   If $F_i(x) \geq F_{[k]}(x)$ $\forall$ x and i = 1,...,k, $F_{[k]}(0)=0$, $F_{[k]} \underset{c}{\prec} G$, $G(x) = 1-e^{-x}$, $x \geq 0$ and $n \geq \max\{r, \frac{r-1}{1-c_1}\}$, then

$$(1.2.15) \qquad \inf_{\Omega} P[CS|R_1] = \int_0^{\infty} H^{k-1}(\frac{x}{c_1})dH(x)$$

where H(x) is the c.d.f. of a χ^2 random variable with 2r d.f.

Proof.   If $G(x) = 1-e^{-x}$ then $a_j = \frac{1}{n}$ for j = 1,2,...,r-1

$$\text{and } a_r = \frac{1}{n}\ (n-r+1).$$

Also $\frac{1}{c_1}\ a_r \geq 1$ \qquad iff \qquad $n \geq \frac{r-1}{1-c_1}$.

By Theorem 1.2.1 and the fact that 2nT is distributed as χ^2 with 2r d.f., the result follows.

If $G(x) = x$ for $0 < x < 1$, then we have the following result which is a special case of Theorem 2.1 of Barlow and Gupta [5].

<u>Corollary 1.2.2.</u>   If $F_i(x) \geq F_{[k]}(x)$ $\forall$ x and i = 1,...,k, $F_{[k]}(0)=0$, $F_{[k]} \underset{c}{\prec} G$ and $G(x) = x$ for $0 < x < 1$,

then

$$(1.2.16) \qquad \inf_{\Omega} P[CS|R_1] = \int_0^1 H_r^{k-1}(\frac{x}{c_1})dH_r(x)$$

where $H_r(x)$ is the c.d.f. of the r-th order statistic from G.

Proof. If $G(x) = x \quad 0 \le x \le 1$ then $a_j = 0$ for $j = 1,\ldots,r-1$ and $a_r = 1$. By Theorem 1.2.1, the result follows.

We state and prove the following theorem about the asymptotic evaluations of the probability of a correct selection.

Theorem 1.2.2. If F_i, $G \subset \mathcal{G}$ for all $i = 1,\ldots,k$ and

  (i) $F_i(x) \ge F_{[k]}(x) \ \forall x$, $i = 1,\ldots,k$, and $F_{[k]} \underset{c}{\prec} G$,

  (ii) $G(x)$ has a differentiable density g in a neighborhood of its α-quantile η_α and $g(\eta_\alpha) \ne 0$,

(iii) $g\,G^{-1}$ is uniformly continuous on $[0,1)$, $G^{-1}(x)$ is convex and there exists an η, $0 < \eta < 1$, such that for $\eta \le y < 1$, $g\,G^{-1}(y)$ is nonincreasing and $\frac{g\,G^{-1}(y)}{1-y}$ is nondecreasing in y, then as $n \to \infty$

$$(1.2.17) \qquad P[CS|R_1] \ge \int_{-\infty}^{\infty} \phi^{k-1}[\frac{x}{c_1} + \frac{1-c}{c_1}\eta_\alpha \ g(\eta_\alpha)(\frac{n}{\alpha\bar\alpha})^{\frac{1}{2}}]d\phi(x)$$

where $\frac{r}{n} \to \alpha$ as $n \to \infty$, $\bar\alpha = 1-\alpha$ and $\phi(x)$ is the standard normal c.d.f.

Proof. See the proof of Theorem 1.2.1, we have

$$(1.2.18) \qquad P[CS|R_1] \ge P[Z_k \ge c_1 \max_{1 \le j < k} Z_j]$$

where $Z_1,\ldots,Z_k$ are i.i.d. with c.d.f. $W_k(x)$ and $W_k(x)$ is the c.d.f. of $T_{(k)}$.

By Theorem 2.2 of Barlow and Van Zwet [11] and (iii),

$(1.2.19)$
$$\sup_{x \geq 0} \left| \int_{F_{[k]}^{-1}(0)}^{x} g[G^{-1}F_{[k]n}(u)]du - \int_{F_{[k]}^{-1}(0)}^{x} g[G^{-1}F_{[k]}(u)]du \right| \to 0 \text{ a.s.}$$

where $F_{[k]n}$ is the empirical distribution of $F_{[k]}$.

Then we have (see Barlow and Doksum [3]), for n large,

$(1.2.20)$
$$Z_i \overset{-}{st} Y_{i;r,n}$$

where $Y_{i;r,n}$ is the r-th order statistic from $H_{F_{[k]}}$ and $H^{-1}_{F_{[k]}}$ (the

inverse of $H_{F_{[k]}}$) is defined in $(1.2.1)$. Since $F_{[k]} \underset{c}{\prec} G$, therefore

$H_{F_{[k]}}$ is convex. Since $G^{-1}(x)$ is increasing and convex, therefore

$G^{-1}H_{F_{[k]}}(x)$ is convex. Since $H_{F_{[k]}} \underset{c}{\prec} G$ and $G^{-1}(0) \leq 0$, therefore

$H_{F_{[k]}} \underset{*}{\prec} G$. In a manner similar to the Theorem 2.1 of Barlow and

Gupta [5], we have

$(1.2.21)$
$$P[Y_{k;r,n} \geq c_1 \max_{1 \leq i \leq k} Y_{i;r,n}] \geq P[Y^*_{k;r,n} \geq c_1 Y^*_{i;r,n}, \ i \neq k]$$

where $Y^*_{i;r,n}$ is the r-th order statistic from G, $i = 1,\ldots,k$. From

$(1.2.18)$, $(1.2.20)$, $(1.2.21)$ and using the fact that

$$Y^*_{i;r,n} \sim N\left(\eta_\alpha, \frac{\alpha\bar{\alpha}}{ng^2(\eta_\alpha)}\right),$$

the theorem follows.

Consistent with the basic probability requirement, we would like

the size of the selected subset to be small. Let S be the size

of the selected subset. One criterion of the efficiency of

the procedure R_1 is the expected value of the size of the subset.

If in addition to the assumptions of the Theorem 1.2.1,

we assume that (a) $F_{[1]}(x) \leq F_{[i]}(x) \geq F_{[1]}(\frac{x}{\delta})$ for all $x \geq 0$,
$i = 1,2,\ldots,k$, where $\delta > 1$ is given and (b) $G \underset{c}{<} F_{[1]}$.
We have the following theorem.

Theorem 1.2.3.

$$(1.2.22) \qquad E(S|R_1) \leq k \int_{G^{-1}(0)}^{\infty} G_T^{k-1}(\frac{\delta}{c_1} x) dG_T(x)$$

where $G_T(x)$ is the c.d.f. of T.

Proof. $E(S|R_1) = \sum_{i=1}^{k} P[T_i \geq c_1 T_j, \text{ for } j \neq i]$.

By Lemma 1.2.6, we have

$$P[T_i \geq c_1 T_j, j \neq i] \leq P[T_k^* \geq c_1 T_j^*, j = 1,\ldots,k-1]$$

where T_j^* has c.d.f. $W_1(x)$ for $j = 1,\ldots,k-1$ and $\frac{T_k^*}{\delta}$ has c.d.f. $W_1(x)$.

So that $E(S|R_1) \leq k P[T_k^* \geq c_1 T_j^*, j = 1,\ldots,k-1]$

$$= k P[Z_k \delta \geq c_1 Z_j, \quad j = 1,\ldots,k-1]$$

where $Z_1,\ldots,Z_k$ are i.i.d. with cdf $W_1(x)$.
Hence

$$E(S|R_1) \leq k P[\frac{\delta}{c_1} Z_k \geq Z_j, j = 1,2,\ldots,k]$$

Let $\varphi(x) = G^{-1} F_{[1]}(x)$

since $G \underset{c}{<} F_{[1]}$ then $-\varphi(x)$ is convex.

Using the same approach as in Theorem 1.2.1

and $\frac{\delta}{c_1} a_r \geq \frac{1}{c_1} a_r$, we have

$$P[\frac{\delta}{c_1} Z_k \geq Z_j, \ j = 1,2,\ldots,k] \leq P[\frac{\delta}{c_1} Z_k^* \geq Z_j^*, \ j = 1,\ldots,k]$$

$$= \int_{G^{-1}(0)}^{\infty} G_T^{k-1}(\frac{\delta}{c_1} x) dG_T(x)$$

where $Z_1^*,\ldots,Z_k^*$ are i.i.d. random variables with c.d.f. $G_T(x)$. This proves the theorem.

Before discussing properties of the selection rule R_1, we give some preliminary definitions. Let Ω denote the set of all k-tuples $\underline{F} = (F_1,\ldots,F_k)$. Let $P_{\underline{F}}(i)$ denote

$$P_{\underline{F}}(i) = P_{\underline{F}}[\pi_{(i)} \text{ is selected}|R]$$

when $\pi_{(i)}$ is associated with $F_{[i]}$.

<u>Definition 1.2.1.</u> A rule R is strongly monotone in $\pi_{(i)}$ if

$P_{\underline{F}}(i)$ is
$\begin{cases} \uparrow & \text{in } F_{[i]} \quad \text{when all other components of } \underline{F} \\ & \text{are fixed} \\ \downarrow & \text{in } F_{[j]}(i \neq j) \quad \text{when all other components of} \\ & \underline{F} \text{ are fixed.} \end{cases}$

<u>Definition 1.2.2.</u> A rule R is monotone means

$$P_{\underline{F}}(i) \leq P_{\underline{F}}(j) \quad \text{for all } \underline{F} \in \Omega \text{ with } F_{[i]}(x) \geq F_{[j]}(x).$$

Definition 1.2.3. A rule R is unbiased if $P_{\underline{F}}(i) \leq P_{\underline{F}}(k)$ for all $\underline{F} \in \Omega$ with $F_{[i]}(x) \doteq F_{[k]}(x)$.

Definition 1.2.4. A rule R is consistent with respect to Ω' means

$\inf\limits_{\Omega'} P[CS|R] \to 1$ as $n \to \infty$.

Theorem 1.2.4. If $a_i \geq 0$ for $i = 1,\ldots,r$, then R_1 is strongly monotone in $\pi_{(i)}$.

Proof.

$$\text{Let } \psi(\underline{x}) = \begin{cases} 1 & \text{if } T_{(i)} \geq c \max\limits_{1 \leq j \leq k} T_{(j)} \\ 0 & \text{otherwise} \end{cases}$$

since $a_i \geq 0$, $T_{(i)}$ is nondecreasing in each of its arguments for $i = 1,\ldots,k$.

So that $\psi(\underline{x})$ is nondecreasing function of $X_{(i)1},\ldots,X_{(i)n}$ when the other components of $\underline{X}$ are held fixed and $\psi(\underline{X})$ is nonincreasing function of $X_{(j)1},\ldots,X_{(j)n}$ $(j \neq i)$ when the other components of $\underline{X}$ are held fixed where $X_{(j)1},\ldots,X_{(j)n}$ are n observations from $F_{[j]}(j = 1,..,k)$. Since $E\psi(\underline{X}) = P_{\underline{F}}(i)$, by Lemma 1.2.3, it follows that R_1 is strongly monotone in $\pi_{(i)}$.

Remark 1.2.1. (1) If a rule R is strongly monotone in $\pi_{(i)}$ for all $i = 1,\ldots,k$, then R is montone and

$$\inf\limits_{\Omega} P[CS|R] = \inf\limits_{\Omega_0} P[CS|R]$$

where $\Omega_0 = \{\underline{F} = (F_1,\ldots,F_k): F_1 = \ldots = F_k\}$.

(2) If R is monotone, then it is unbiased.

(3) If $F_i(x) = F(x,\theta_i)$, $i=1,\ldots,k$ and T_i is a consistent estimator of θ_i, then R_1 is consistent.

(4) If $F_i, G \in \mathfrak{F}$, $F_i \underset{c}{\leq} G$, $i=1,\ldots,k$ and the condition (iii) (excluding $G^{-1}(x)$ is convex) of Theorem 1.2.2 is satisfied, we can show that R_1 is consistent.

The selection of the population with largest F_i $(i=1,..,k)$ can be handled analogously. We assume $F_{[i]}(x) \leq F_{[1]}(x)$, $i=1,...,k$, and $F_{[1]} \underset{c}{\prec} G$. The rule for selecting the population with $F_{[1]}$ is

R_2: Select population π_i if and only if

(1.2.23) $c_2 T_i \leq \min_{1 \leq j \leq k} T_j$

where $c_2(0 < c_2 \leq 1)$ is determined so as to satisfy the basic requirement. In a manner similar to the proof of Theorem 1.2.1, if F_i, $G \in \mathfrak{d}$, $F_{[i]}(x) \leq F_{[1]}(x)$ $\forall x$ and $i = 1,...,k$, $F_{[1]}(0) = 0$ and $F_{[1]} \underset{c}{\prec} G$ and if $a_j \geq 0$ for $j = 1,...,r$, $G^{-1}(0) \leq 0$, $g\, G^{-1}(0) \leq 1$ and $a_r \geq c_2$, then

(1.2.24) $P[CS|R_2] \geq \int_{G^{-1}(0)}^{\infty} \bar{G}_T^{k-1}(c_2 x)dG_T(x)$

where $\bar{G}_T(x) = 1-G_T(x)$.

(B) <u>Efficiency of procedure R_1 under slippage configuration</u>.

We consider slippage configuration $F_{[i]}(x) = F(\frac{x}{\delta})$, $i = 1,2,..,k-1$, and $F_{[k]}(x) = F(x)$, $0 < \delta < 1$. Let $E(S|R)$ denote the expected subset size using the rule R. Then $E(S|R)-P[CS|R]$ is the expected number of non-best populations included in the selected subset. For a given $\epsilon > 0$, let $n_R(\epsilon)$ be the asymptotic sample size for which $E(S|R)-P[CS|R] = \epsilon$. We define the asymptotic relative efficiency $A R E(R,R^*,\delta)$ of R relative to R^* to be the limit as $\epsilon \to 0$ of the

ratio $\frac{n_R(\epsilon)}{n_{R\star}(\epsilon)}$ i.e. ARE $(R,R\star;\delta) = \lim\limits_{\epsilon \to 0} \frac{n_R(\epsilon)}{n_{R\star}(\epsilon)}$.

Under the slippage configuration, for large n, we have (see Theorem 1.2.3)

(1.2.25) $E(S|R_1) = P[CS|R_1] + (k-1)P[T_{(1)} \geq c_1 \max\limits_{i \neq 1} T_{(i)}]$

(1.2.26) $P[T_{(1)} \geq c \max\limits_{i \neq 1} T_{(i)}] \approx P[Y_1 \geq c_1 \max\limits_{i \neq 1} Y_i]$

where $Y_1,\ldots,Y_k$ are independent and Y_i is the r-th order statistic from $H_{F_{[i]}}$ for $i = 1,\ldots,k$. The right-hand side of (1.2.26) is equal to

(1.2.27) $\int\limits_{-\infty}^{\infty} \phi(\frac{\delta x}{c_1} - a_\alpha h(a_\alpha)(1 - \frac{\delta}{c_1})(\frac{n}{\alpha\alpha})^{\frac{1}{2}})\cdot$

$\phi^{k-2}(\frac{x}{c_1} - a_\alpha h(a_\alpha)(1 - \frac{1}{c_1})(\frac{n}{\alpha\alpha})^{\frac{1}{2}})d\phi(x)$

where c_1 is the constant used in defining R_1, a_α is the (unique) α-quantile of $H_{F_{[k]}}(x)$ and $h(x)$ is the density function of $H_{F_{[k]}}(x)$.

From now on, it is assumed that $k = 2$. Therefore (1.2.25) reduces to

(1.2.28) $E(S|R_1) - P[CS|R_1] = \phi(-h(a_\alpha)a_\alpha(1 - \frac{\delta}{c_1})(\frac{n}{\alpha\alpha})^{\frac{1}{2}}(1 + \frac{\delta^2}{c_1^2})^{-\frac{1}{2}})$.

Let $\int\limits_{-\infty}^{\infty} \phi^{k-1}(\frac{x}{c_1} + (1-c_1)n_\alpha g(n_\alpha)\frac{1}{c_1}(\frac{n}{\alpha\alpha})^{\frac{1}{2}})d\phi(x) = P\star$.

Barlow and Gupta [5] have proved that

$$c_1 = 1 - \frac{2^{\frac{1}{2}}D}{n^{\frac{1}{2}}} + \frac{D}{n} - \frac{3}{2^{3/2}}\frac{D^3}{n^{3/2}} + O(\frac{1}{n^2})$$

where $D = \frac{\phi^{-1}(P\star)(\alpha\bar{\alpha})^{\frac{1}{2}}}{n_\alpha g(n_\alpha)}$.

Now, setting the right hand side of (1.2.28) equal to ϵ and using

$c_1 \approx 1 - \dfrac{2^{\frac{1}{2}}D}{n^{\frac{1}{2}}}$, we obtain

(1.2.29) $n_{R_1}(\epsilon) \approx [-(a\bar{a})^{\frac{1}{2}}\phi^{-1}(\epsilon)(1+\epsilon^2)^{\frac{1}{2}}+\sqrt{2}\ D\delta a_\alpha h(a_\alpha)]^2 \cdot$

$$[a_\alpha^2 h^2(a_\alpha)(1-\delta)^2]^{-1}.$$

Comparison with Barlow-Gupta Procedure

Barlow and Gupta [5] propose a procedure R_3, for a quantile selection problem.

R_3: Select population π_i if and only if

(1.2.30) $$T_{r,i} \geq c_3 \max_{1 \leq j \leq k} T_{r,j}$$

where c_3 $(0 < c < 1)$ is chosen to satisfy $P[CS|R] \geq P^*$ and $T_{r,i}$ is the r-th order statistic from F_i where $r \leq (n+1)\alpha < r+1$. A similar expression for $n_{R_3}(\epsilon)$ (see Barlow and Gupta [5]) is

$$n_{R_3}(\epsilon) \approx [-(a\bar{a})^{\frac{1}{2}}\phi^{-1}(\epsilon)\ (1+\epsilon^2)^{\frac{1}{2}}+\sqrt{2}\ D\delta\xi_\alpha f(\xi_\alpha)]^2 [\xi_\alpha^2 f^2(\xi_\alpha)(1-\delta)^2]^{-1}$$

where f is the density of F with unique α-quantile, ξ_α.

(1.2.31) $\mathrm{ARE}(R_1,R_3;\delta) = \lim_{\epsilon \to 0} \dfrac{n_{R_1}(\epsilon)}{n_{R_3}(\epsilon)} = \dfrac{\xi_\alpha^2\ f^2(\xi_\alpha)}{a_\alpha^2\ h^2(a_\alpha)}$.

If $G(x) = 1-e^{-x}$, $x > 0$ and $F(x) = 1-e^{-x}$, $x > 0$, then

(1.2.32) $\mathrm{ARE}(R_1,R_3;\delta) = \dfrac{(1-\alpha)^2\log^2(1-\alpha)}{\alpha^2} \leq 1.$

$$= 0.4803, \quad \alpha = \frac{1}{2}.$$

Comparison with Gupta Procedure

Gupta [30] gave a selection procedure for gamma populations

π_i's with densities $\dfrac{1}{\Gamma(a)\theta_i^r}\, x^{a-1}\, e^{-\frac{x}{\theta_i}}$ $x > 0$, $\theta_i > 0$, $i = 1,2,\ldots,k$.

The procedure R_4 is

R_4:   Select population π_i if and only if

$$(1.2.33) \qquad \bar{X}_i \geq c_4 \max_{1 \leq j \leq k} \bar{X}_j$$

where $\bar{X}_i$ is the sample mean of size n from π_i and c_4 is the largest

constant $(0 < c_4 \leq 1)$ chosen so that $P[CS|R_4] \geq P^*$.

For $k = 2$, $\theta_{[1]} = \cdot$ and $\theta_{[2]} = 1$, Barlow and Gupta [5] have prove that

$$(1.2.34) \qquad ARE(R_3,R_4;\cdot) = \lim_{\varepsilon \to 0} \frac{n_{R_3}(\varepsilon)}{n_{R_4}(\varepsilon)}$$

$$= \frac{a(\log \cdot)^2\, a\bar{\alpha}(1 + \delta^2)}{2(1-\cdot)^2 [\cdot_t\, f(\cdot_t)]^2}\ .$$

It is easy to show that

$$(1.2.35)\quad ARE(R_1,R_4;\delta) = ARE(R_1,R_3;\cdot)ARE(R_3,R_4;\delta)$$

$$= \{\frac{\sqrt{a}\ \log \cdot}{\sqrt{2}}\ \frac{\sqrt{a\bar{\alpha}}\ \sqrt{1+\delta^2}}{(1-\cdot)a_t\, h(a_t)}\}2\ .$$

If $G(x) = 1-e^{-x}$ for $x > 0$ and $a = 1$,

then

$$(1.2.36)\quad ARE(R_1,R_4;\delta) = \frac{(1-\iota)(1+\delta^2)\log^2\delta}{2(1-\delta)^2\alpha}\ .$$

Therefore as $\delta \to 1$, we get

(1.2.37) $ARE(R_1, R_4; \delta+1) = \frac{1-\alpha}{\alpha}$.

It is easy to see that

(1.2.38) $ARE(R_1, R_4; \delta+1) \begin{cases} < 1, & \alpha > \frac{1}{2} . \\ = 1, & \alpha = \frac{1}{2} . \\ > 1, & \alpha < \frac{1}{2} . \end{cases}$

Comparison of R_1 and R_5 from uniform distribution

Suppose π_1 and π_2 are two independent uniform populations with distribution functions F_i $(i = 1,2)$.

$$F_i(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{\theta_i} & 0 \le x \le \theta_i \\ 1 & x > \theta_i \end{cases}$$

where $\delta = \theta_{[1]} < \theta_{[2]} = 1$.

A sample size n is drawn from each of the two populations. Let T_i^* be the total life statistic until r-th failure from π_i $(i = 1,2)$ where $r \le (n+1)\alpha < r+1$.

The procedure R_5 is given by

R_5: Select population π_i if and only if

(1.2.39)
$$T_i^* \geq c_5 \max_{1 \leq j \leq k} T_j^*$$

where c_5 is chosen so that $P[CS|R_5] \geq P^*$. Let $T_{(i)}^*$ be associated with $\pi_{[i]}$.

(1.2.40) $E(S|R_5) - P[CS|R_5] = P[T_{(1)}^* \geq c_5 T_{(2)}^*] = P[T_1' \geq \frac{c_5}{\delta} T_2']$

where T_1', T_2' are two independent total life statistic until r-th failure from uniform distribution over $(0,1)$. By Gupta and Sobel [42],

(1.2.41)
$$\frac{T_i' - u}{\sigma} \to N(0,1) \qquad \text{as } n \to \infty,$$

where $u = \frac{n\alpha(2n - \alpha n + 1)}{2n+1} \approx u' = \frac{n\alpha(2-\alpha)}{2}$

$$\sigma^2 = \frac{\alpha(1-\alpha)n^2(2-\alpha)^2 n^2}{4n^3} + \frac{\alpha^3 n^3}{12n^2}$$

$$= A n \qquad \text{where } A = \frac{\alpha(1-\alpha)(2-\alpha)^2}{4} + \frac{\alpha^3}{12} .$$

Hence $\frac{u}{\sigma} \approx \frac{u'}{\sigma} = B\sqrt{n}$ where $B = \frac{\alpha(2-\alpha)}{2\sqrt{A}}$.

From (1.2.40), we have

$$E(S|R_5) - P[CS|R_5] = P[\frac{T_1' - u'}{\sigma} \geq \frac{c_5}{\delta} (\frac{T_2' - u'}{\sigma}) + (\frac{c_5}{\delta} - 1) \frac{u'}{\sigma}]$$

$$\approx P[Z_1 \geq \frac{c_5}{\delta} Z_2 + (\frac{c_5}{\delta} - 1)B\sqrt{n}]$$

where Z_1, Z_2 are i.i.d. with $N(0,1)$.

Hence

$$E(S|R_5) - P[CS|R_5] = \int_{-\infty}^{\infty} \phi[\frac{\delta}{c_5} x - (1 - \frac{\delta}{c_5})B\sqrt{n}] d\Phi(x)$$

$$= \phi\left[\frac{-(1 - \frac{\delta}{c_5})B\sqrt{n}}{\sqrt{1+(\frac{\delta}{c_5})^2}}\right].$$

Let $E(S|R_5) - P[CS|R_5] = \epsilon > 0$, we obtain

$$(1.2.42) \qquad (\frac{1}{c_5} - \frac{1}{\delta})\sqrt{n} = \sqrt{\frac{1}{\delta^2} + \frac{1}{c_5^2}} \cdot \frac{\phi^{-1}(\epsilon)}{B}$$

Note that

$$\inf_{\Omega} P(CS|R_5) = P[T_1' \geq c_5 T_2']$$

where T_1', T_2' are defined as above.

$$P[T_1' \geq cT_2'] = P[\frac{T_1'-u'}{\sigma} \geq c_5[\frac{T_2'-u'}{\sigma}] + (c_5-1)\frac{u'}{\sigma}]$$

$$\approx P[Z_1 \geq c_5 Z_2 + (c_5-1)B\sqrt{n}]$$

where Z_1, Z_2 are i.i.d. with $N(0,1)$.

Hence

$$P[T_1' \geq cT_2'] = \int_{-\infty}^{\infty} \phi(\frac{1}{c_5} x - (1 - \frac{1}{c_5})B\sqrt{n})d\phi(x)$$

$$= \phi\left[\frac{-(1 - \frac{1}{c_5})B\sqrt{n}}{\sqrt{1 + \frac{1}{c_5^2}}}\right]$$

Setting $\inf_{\Omega} P[CS|R_5] = P^*$, we obtain

$$-(1 - \frac{1}{c_5})B\sqrt{n} = \phi^{-1}(P^*)\sqrt{1 + \frac{1}{c_5^2}}$$

$$(1-c_5)\sqrt{n} = D\sqrt{1 + c_5^2} \quad \text{where } D = \frac{\phi^{-1}(P^*)}{B}.$$

We see that $c_5 \approx 1- \frac{\sqrt{2}\,D}{\sqrt{n}}$,

$$\frac{1}{c_5} \approx 1 + \frac{\sqrt{2}\,D}{\sqrt{n}} .$$

From (1.2.42),

$$(1 + \frac{\sqrt{2}\,D}{\sqrt{n}} - \frac{1}{\delta})\sqrt{n} = \frac{\phi^{-1}(\epsilon)}{B}\,\{\frac{1}{\delta^2} + [1 + \frac{2\sqrt{2}\,D}{\sqrt{n}} + \frac{2D^2}{n}]\}^{\frac{1}{2}}$$

$$\sqrt{n}(1 - \frac{1}{\delta}) + \sqrt{2}\,D \approx \frac{\phi^{-1}(\epsilon)}{B}\,\{\frac{1}{\delta^2} + 1\}^{\frac{1}{2}}$$

Thus

$$(1.2.43) \qquad n_{R_5}(\epsilon) \approx \{\frac{\phi^{-1}(\epsilon)\sqrt{1+\delta^2} - \sqrt{2}\,\delta\phi^{-1}(P\star)}{B(1-\delta)}\}^2 .$$

If we assume that $G(x) = x$ for $0 < x < 1$, then the $n_{R_1}(\epsilon)$ is given by (1.2.29) with $a_\alpha = \alpha$, $h(a_\alpha) = 1$, $n_\alpha = \alpha$ and $g(n_\alpha) = 1$. Hence

$$(1.2.44) \qquad ARE(R_1,R_5;\delta) = \lim_{\epsilon \to 0} \frac{n_{R_1}(\epsilon)}{n_{R_5}(\epsilon)}$$

$$= \frac{B^2(1-\alpha)}{\alpha} = \frac{3(1-\alpha)(2-\alpha)^2}{3(1-\alpha)(2-\alpha)^2+\alpha^2} < 1$$

$$= 0.931, \; \alpha = \frac{1}{2}.$$

$$= 0.675, \; \alpha = \frac{3}{4}.$$

$$= 0.574, \; \alpha = \frac{4}{5}.$$

(C) <u>Selection procedure for distribution $\underset{c}{\prec}$ ordered with respect to Weibull distribution</u>

Assumed that the specified distribution $G(x)$ is a Weibull distribution. In other words, $G(x)$ is given by

$$G(x) = \begin{cases} 1-e^{-\lambda x^\alpha} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

where $\lambda > 0$ and attention is restricted to $\alpha \geq 1$ which is assumed known.

In this case, we use the statistic T_i^* which is defined by

$$(1.2.45) \qquad T_i^* = \sum_{j=1}^{r} X_{i;j,n}^{\alpha} + (n-r)X_{i;r,n}^{\alpha}, \quad i = 1,\ldots,k,$$

(as before, $X_{i;j,n}$ denote the j-th order statistic from F_i, $i=1,\ldots,k$). Since $G(x)$ is convex with respect to the exponential distribution if $\alpha \geq 1$ and since the convex ordering is transitive, the family of distributions which are convex with respect to Weibull ($\alpha \geq 1$) will have IFR distribution. Thus our interest here is in a special subclass of IFR distributions.

The rule for selecting the population which is associated with $F_{[k]}$ is as follows,

R_6: Select population π_i if and only if

$$(1.2.46) \qquad T_i^* \geq c_6 \max_{1 \leq j \leq k} T_j^*$$

where c_6 $(0 < c_6 \leq 1)$ is determined so as to satisfy the basic probability requirement.

Using the fact that if $F \underset{c}{\prec} G$ and $F(0) = G(0) = 0$ then

$$F_\alpha \underset{c}{\prec} G_\alpha \quad \text{for } \alpha \geq 1,$$

where F_α is the c.d.f. of X^α, $F(x)$ is the c.d.f. of X, G_α is the c.d.f. of Y^α and $G(y)$ is the c.d.f. of Y.

Also,

$$G_\alpha^{-1} F_\alpha(X_{i,n}^\alpha) \underset{st}{} \text{ i-th order statistic from } G^*(x) = 1-e^{-\lambda x},$$

$x \geq 0$ where $X_{1,n} \leq \ldots \leq X_{n,n}$ are order statistics from F. In a

manner similar to the proof of Theorem 1.2.1, one can prove the following theorem.

Theorem 1.2.3. If $F_i(x) \geq F_{[k]}(x) \ \forall \ x$ and $i = 1,\ldots,k$,

$F_{[k]}(0) = 0$, $F_{[k]} \underset{c}{\prec} G$, $G(x) = 1-e^{-\lambda x^\alpha}$, $x > 0$, $\lambda > 0$ and $\alpha(\geq 1)$

is known. If $n \geq \max\{r, \frac{r-1}{1-c_6}\}$, then

$$(1.2.47) \quad \inf P[CS|R_6] = \int_0^\infty G_T^{k-1}(\frac{x}{c_6})dG_T(x)$$

where $G_T(x)$ is the c.d.f. of a χ^2 random variable with 2r d.f.

(D) **Selection with respect to the means for Gamma populations**

Let $\pi_1,\ldots,\pi_k$ be k populations with densities

$$r_i(x) = \frac{\beta^{\alpha_i}}{\Gamma(\alpha_i)} x^{\alpha_i-1} e^{-\beta x}, \ x \geq 0, \ \beta > 0, \ \alpha_i \geq 1, \ i = 1,\ldots,k.$$

Let $R_i(x)$ be the distribution function of π_i, $i = 1,\ldots,k$. We are given a sample of size n from each π_i. Let T_i^* be total life statistic until r-th failure from π_i.

Let $\alpha_{[1]} \leq \cdots \leq \alpha_{[k]}$ be the ordered values of α_i's. We are interested in selecting the population with the largest value $\alpha_{[k]}$ (unknown). Since the mean of π_i is $\frac{\alpha_i}{\beta}$, selection of the population with largest mean is equivalent to selecting the population with largest value, $\alpha_{[k]}$. The subset selection rule based on T_i is:

R_7: Select population π_i if and only if

(1.2.48) $$T_i^* \geq c_7 \max_{1 \leq j \leq k} T_j^*,$$

where c_7 $(0 < c_7 < 1)$ is chosen to satisfy

$$P[CS|R_7] \geq P^*.$$

Since the rule R_7 is scale invariant, we can assume $\beta = 1$.

Case 1: All α_i are unknown and ≥ 1. Let $\Omega_1 = \{\underline{\alpha} = (\alpha_1,\ldots,\alpha_k): \alpha_i \geq 1 \; \forall \; i\}$

$$R_i \underset{c}{\prec} G(x) = 1-e^{-x}, \quad x > 0, \; i = 1,\ldots,k.$$

In this case, by Corollary 1.2.2, we have the following result.

If $n \geq \max\{r, \frac{r-1}{1-c}\}$, then $\inf_{\Omega_1} P[CS|R_7] = \int_0^\infty H^{k-1}(\frac{x}{c})dH(x)$,

where $H(x)$ is the c.d.f. of a χ^2 r.v. with 2r d.f.

Case 2: α_i are unknown but assume $1 \leq \alpha_i \leq \Delta$, $i = 1,..,k$ and Δ is known.

Let $R_\Delta(x)$ be the c.d.f. of X with density function

$$r_\Delta(x) = \frac{\beta^\Delta}{\Gamma(\Delta)} x^{\Delta-1}e^{-\beta x}, \quad x > 0, \; \beta > 0.$$

Let $h(x)$ be the density function of a χ^2 r.v. with 2r d.f.

<u>Theorem 1.2.4.</u>

(1.2.49) $P[CS|R_7] \geq \int_0^\infty H^{k-1}(\frac{2n}{c_7} x) \frac{2nh(2ny)}{r_\Delta(y)} e^{-x} dx$,

where $y = R_\Delta^{-1} (1-e^{-x})$.

Proof. $P[CS|R_7] = P[T_{(k)}^* \geq c_7 \max_{1 \leq j \leq k-1} T_{(j)}^*]$,

where $T_{(i)}^*$ is associated with $\alpha_{[i]}$, $i = 1,\ldots,k$.

Since $R_\Delta(x) \leq R_i(x) \leq G(x) = 1-e^{-x}$, by Lemma 1.2.6, then

$$(1.2.50) \qquad P[CS|R_7] \geq P[T_k^{**} \geq c_7 \max_{1 \leq j \leq k-1} T_j^{**}]$$

where T_k^{**} is the total life statistic until r-th failure from $G(x)$

and $T_j^{**}(j = 1,\ldots,k-1)$ is the total life statistic until r-th

failure from $R_\Delta(x)$.

Since $\Delta \geq 1$ then $R_\Delta \underset{c}{\prec} G$.

Let $\varphi(x) = G^{-1}R_\Delta(x)$.

$$(1.2.51) \qquad P[T_k^{**} \geq c_7 T_j^{**}, j=1,..,k-1] = P[\varphi(\frac{1}{n} T_k^{**}) \geq \varphi(\frac{c_7}{n} T_j^{**})\ j=1,\ldots,k-1]$$

By Lemma 1.2.4 (a) with $a_1 = \ldots = a_{r-1} = \frac{c_7}{n}$, $a_r = \frac{(n-r+1)c_7}{n}$

$$a_i = 0 \quad \text{for } i \geq r+1$$

and $\varphi(x) \underset{st}{=} Y$ where X is a r.v. with distribution R_Δ

Y is a r.v. with distribution G,

we have

$$(1.2.52) \qquad P[\varphi(\frac{1}{n} T_k^{**}) \geq \varphi(\frac{c_7}{n} T_j^{**}), j=1,\ldots,k-1] \geq P[\varphi(\frac{1}{n} T_k^{**}) \geq \frac{c_7}{2n} Y_j,$$

$$j = 1,\ldots,k-1]$$

where Y_j (j = 1,\ldots,k-1) is a r.v. with χ^2 with 2r d.f. From

(1.2.50), (1.2.51) and (1.2.52), we have

$$P[CS|R_7] \geq \int_0^\infty H^{k-1}(\frac{2n}{c_7} x)dB(x),$$

where $B(x) = P[\varphi(\frac{1}{n} T_k^{**}) \leq x]$.

Since $\varphi(x) = -\ell n (1-R_\Delta(x))$, then $\varphi^{-1}(x) = R_\Delta^{-1}(1-e^{-x})$.

Thus

$$B(x) = P[T_k^{\star\star} \leq n\ \varphi^{-1}(x)] = H(2n\ \varphi^{-1}(x))$$

$$= H[2n\ R_\Delta^{-1}(1-e^{-x})].$$

Now

$$\frac{dB(x)}{dx} = h[2ny]\ \frac{d(2ny)}{dx}$$

$$= h(2ny)2n \cdot \frac{e^{-x}}{r_\Delta(y)}\quad \text{where } y = R_\Delta^{-1}(1-e^{-x}).$$

So that,

$$\int_0^\infty H^{k-1}(\frac{2n}{c_7}x)dB(x) = \int_0^\infty H^{k-1}(\frac{2n}{c_7}x)\ \frac{2nh(2ny)}{r_\Delta(y)}\ e^{-x}dx.$$

This completes the proof.

Let S denote the size of the selected subset. The expected value of S when R_7 is used is given by

$$(1.2.53)\quad E(S|R_7) = \sum_{i=1}^{k} P[T_i^\star \geq c_7 \max_{1\leq j\leq k} T_j^\star].$$

Let $\Omega' = \{\underline{\alpha} = (\alpha_1,\ldots,\alpha_k): 1 \leq \alpha_i \leq \Delta,\ i = 1,\ldots,k\}.$

For $\underline{\alpha} \in \Omega'$, since $R_\Delta(x) \leq R_i(x) \leq G(x) = 1-e^{-x}$, then

$$E(S|R_7) \leq \sum_{i=1}^{k} P[T_1^{\star\star} \geq c_7 \max_{2\leq j\leq k} T_j^{\star\star}]$$

where $T_1^{\star\star}$ is the total life statistic until r-th failure from $R_\Delta(x)$ and $T_j^{\star\star}(j = 2,\ldots,k)$ is the total life statistic until r-th failure from $G(x)$.

Hence $\quad E(S|R_7) \leq k\ P[T_1^{\star\star} \geq c_7 \max_{2\leq j\leq k} T_j^{\star\star}]$

so that,

$$(1.2.54) \qquad \sup_{\underline{\omega}'} E(S|R_7) = k \int R^{k-1}(\frac{x}{c_7}) dS(x)$$

where $R(x)$ $(S(x))$ is the c.d.f. of total life statistic until r-th failure from $G(x)$ $(R_\Delta(x))$.

1.3. Selecting a best population - using indifference zone approach.

Let $\pi_1,\ldots,\pi_k$ be k populations. The random variable X_i associated with π_i has an absolutely continuous distribution F_i. We assume there exists a $F_{[k]}(x)$ such that $F_{[i]}(x) \geq F_{[k]}(\frac{x}{\delta})$ for all x, i=1,...k-1 and δ $(0 < \delta \leq 1)$ is specified. Let

$$(1.3.1) \qquad \Omega = \{\underline{F} = (F_1,\ldots,F_k): \exists\, a_j \text{ such that}$$

$$F_i(x) \geq F_j(\frac{x}{\delta}) \quad \forall\ i \neq j\} \ .$$

The correct selection is the choice of any population which is associated with $F_{[k]}$. We propose the selection rule R_8: Select population π_i if and only if

$$(1.3.2) \qquad T_i = \max_{1 \leq j \leq k} T_j \qquad \text{where } T_i \text{ is defined as in Section 1.2.}$$

We want the $P[CS|R_8] \geq P^*$, for all $\underline{F} \in \Omega$, where P^* $(\frac{1}{k} < P^* < 1)$ is specified.

Theorem 1.3.1. If F_i, $G \in \mathfrak{I}$, i = 1,...,k, $F_{[k]}(0) = 0$, $F_{[k]} \underset{c}{\leq} G$ and $G^{-1}(0) \leq 0$. If $a_j \geq 0$, j = 1,...,r,

$$gG^{-1}(0) \leq 1 \text{ and } a_r \geq \delta,$$

then

(1.3.3) $$P[CS|R_8] \geq \int_0^\infty G_T^{k-1}(\frac{x}{\delta})dG_T(x)$$

where $G_T(x)$ is the c.d.f. of T.

Proof. $P[CS|R_8] = P[T_{(k)} \geq \max_{1 \leq j \leq k} T_{(j)}]$.

Since $F_{[i]}(\delta x) \geq F_{[k]}(x)$, $i=1,\ldots,k-1$, it follows from Lemma 1.2.6 that

$$P[CS|R_8] = P[T_{(k)} \geq \delta \frac{T_{(j)}}{\delta} \; \forall j \neq k]$$

$$\geq P[T_{(k)} \geq \delta \; T_j^* \; \forall j \neq k]$$

where $T_1^*,\ldots,T_{j-1}^*$, $T_{(k)}$ are i.i.d. with c.d.f. $W_k(x)$. Using the same argument as in Theorem 1.2.1, we have our theorem.

<u>Remark 1.3.1.</u> $\inf_\Omega P[CS|R] = \int_{G^{-1}(0)}^\infty G_T^{k-1}(\frac{x}{\delta})dG_T(x)$ if $G^{-1}(0) = 0$.

For given k, δ, P* and G(x), we can possibly find the values of the pair (n,r), (n $\geq$ r) which satisfy

(1.3.4) $$a_r \geq \delta \text{ and } \int_{G^{-1}(0)}^\infty G_T^{k-1}(\frac{x}{\delta})dG_T(x) \geq P*.$$

If G(x) = x for 0 < x < 1, we can always find the values of the pair (n,r), (n $\geq$ r) which satisfy

$$\int_0^\infty G_T^{k-1}(\frac{x}{\delta})dG_T(x) \geq P*.$$

In this case, $G_T(x)$ is the c.d.f. of r-th order statistic based on a sample of size n from G.

If $G(x) = 1-e^{-x}$ for $x \geq 0$ (see Patel [58]), we can find the smallest integer r which satisfies

$\int_0^\infty G_T^{k-1}(\frac{x}{\delta})dG_T(x) \geq P^*$ where $G_T(x)$ is the c.d.f. of a χ^2 random variable with 2r d.f. Since $\frac{1}{\delta} a_r \geq 1$ iff $n \geq \frac{r-1}{1-\delta}$, the minimum n

satisfies $n \geq \max\{r, \frac{r-1}{1-\delta}\}$.

1.4. Selection procedure for distribution $\underset{*}{\prec}$ ordered with respect to G.

Let $t(> 0)$ be a given number. Let $N_i(t)$ be the number of failures in time t among the n units on life test from π_i which has a continuous distribution F_i, $i = 1,...,k$. We assume that

(1.4.1) $N_i(t) \geq 1$, $i = 1,...,k$,

and there exists a $F_{[k]}(x)$ such that $F_i(x) \geq F_{[k]}(x)$ for all x, $i = 1,...,k$. The correct selection is the choice of any population which is associated with $F_{[k]}$. Let T_i be the total life statistic until $N_i(t)$-th failure from population π_i. Let $T_{(i)}$ be associated with $F_{[i]}$. We propose the rule

(1.4.2) R_9: Select $\pi_i \Leftrightarrow T_i \geq c_9 \max_{1\leq j\leq k} T_j$

where $c_9(0 < c_9 \leq 1)$ is chosen so that $P[CS|R_9] \geq P^*$.

Theorem 1.4.1. If $F_i(x) \geq F_{[k]}(x)$ $\forall$ x, $i = 1,...,k$ and $F_{[k]} \underset{*}{\prec} G$, then

(1.4.3) $P[CS|R_9] \geq \int_0^\infty A_1^{k-1}(\frac{x}{c_9})dA_2(x)$

where $A_1(x)(A_2(x))$ is the c.d.f. of total life statistic until n-th (first) failure from G.

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Proof. From (1.4.1), we have

$$P[CS|R_9] = P[T_{(k)} \geq c_9 \max T_{(j)}]$$

$$\geq P[T_k^\star \geq c_9 T_j^\star \text{ for } j = 1,\ldots,k-1]$$

where $\dfrac{T_i^\star}{n}$ is the mean of a sample of size n from $F_{[i]}$, $i=1,\ldots,k-1$

and $\dfrac{T_k^\star}{n}$ is the first order statistic from $F_{[k]}$.

Let

$$(1.4.4) \qquad T_j^{\star\star} = \sum_{\ell=1}^{n} X_{j\ell} \qquad \text{for } j = 1,\ldots,k-1.$$

where $X_{j1},\ldots,X_{jn}$ are i.i.d. from $F_{[k]}$, $j = 1,\ldots,k-1$.

sinc $F_{[j]}(x) \geq F_{[k]}(x)$ then $T_j^{\star\star} \underset{st}{\geq} T_j^\star$. Hence

$$(1.4.5) \qquad P[T_k^\star \geq c_0 T_j^\star, j \neq k] \geq P[T_k^\star \geq c_9 T_j^{\star\star}, j \neq k].$$

Let $\varphi(x) = G^{-1} F_{[k]}(x)$.

$$(1.4.6) \qquad P[T_k^\star \geq c_9 T_j^{\star\star}, j \neq k] = P[\varphi(\dfrac{T_k^\star}{nc_9}) \geq \varphi(\dfrac{1}{n} T_j^{\star\star}), j \neq k]$$

Since φ is starshaped, then

$$(1.4.7) \qquad \varphi(\dfrac{1}{c_9} \dfrac{T_k^\star}{n}) \geq \dfrac{1}{c_9} \varphi(\dfrac{T_k^\star}{n}).$$

By Lemma 1.2.5 (a) and (1.4.4),

$$(1.4.8) \qquad \varphi(\dfrac{1}{n} T_j^{\star\star}) \leq \sum_{\ell=1}^{n} \dfrac{1}{n} \varphi(X_{j\ell}).$$

Note that $\varphi(\dfrac{T_k^\star}{n}) \underset{st}{=} Y_k$ where Y_k is the first order statistic of size n from G and $\varphi(X_{j\ell})$ has distribution G, $j = 1,\ldots,k-1$, $\ell = 1,\ldots,n$.

Let $Y_j = \sum_{\ell=1}^{n} \varphi(X_{j\ell})$, $j = 1,..,k-1$.

From (1.4.7) and (1.4.8), the right-hand side of (1.4.6) is greater than or equal to

$$P[\frac{1}{c_9} Y_k \geq \frac{1}{n} Y_j, \ j=1,\ldots,k-1] = P[\frac{1}{c_9}(nY_k) \geq Y_j, \ j=1,\ldots,k-1]$$

$$= \int_0^{\infty} A_1^{k-1}(\frac{x}{c_9}) dA_2(x).$$

The proof is complete.

1.5. Selection with respect to the means for distributions s-ordered with respect to a specified distribution G.

Let the random variable X_i associated with π_i have an absolutely continuous distribution F_i. Assume that $F_i(x) = F(\frac{x}{\theta_i})$, $i=1,\ldots,k$, $F \underset{s}{<} G$ and F and G are symmetric about 0. As before, let $0 < \theta_{[1]} \leq \cdots \leq \theta_{[k]}$ be ordered values of unknown θ_i $(i = 1,\ldots,k)$. A best population is defined as the one having $\theta_{[k]}$. We are given a sample of size n from each of the k populations. Assume that n is large. Let

(1.5.1) $T_i = -r'X_{i;r',n} - X_{i;r'+1,n} - \cdots - X_{i,\ell_i,n} + X_{i;\ell_i+1,n} + \cdots$

$+ (n-r+1)X_{i;r,n}.$

where $X_{i;1,n} \leq \cdots \leq X_{i;\ell_i,n} < 0 \leq X_{i;\ell_i+1,n} \leq \cdots \leq X_{i;n,n}$ are order statistics from $F_i(x)$, and r',r are given integers such that $1 \leq r' \leq \ell_i$, $n \geq r > \ell_i$, $i = 1,\ldots,k$.
Similarly, let

$$(1.5.2) \qquad T = -r'Y_{r',n} - Y_{r'+1,n} - \cdots - Y_{\ell,n} + Y_{\ell+1,n} + \ldots + (n-r+1)Y_{r,n}$$

where $Y_{1,n} \leq \cdots \leq Y_{\ell,n} < 0 < Y_{\ell+1,n} \leq \cdots \leq Y_{n,n}$ are order statistics from G. Then, for selecting a subset containing $\theta_{[k]}$, we propose the following selection rule:

R_{10}: Select population π_i if and only if

$$(1.5.3) \qquad T_i \geq c_{10} \max_{1 \leq j \leq k} T_j$$

where c_{10} $(0 < c_{10} < 1)$ is determined as to satisfy the probability requirement, inf $P[CS|R_{10}] \geq P^*$, where CS stands for a correct selection, i.e., the selection of any subset which contains the population with $\theta_{[k]}$ and $\Omega = \{(\theta_1, \ldots, \theta_k): \theta_i > 0, i=1, \ldots, k\}$.

Let $\pi_{(i)}$ be associated with $\theta_{[i]}$, $i = 1, \ldots, k$. Now, we give a theorem pertaining to the computation of the minimum probability of a correct selection.

Theorem 1.5.1. If $F_i(x) = F(\frac{x}{\theta_i})$ $\forall x$ and $i = 1, \ldots, k$, $F \underset{s}{<} G$ and F and G are symmetric about 0.

If $\lim\limits_{n \to \infty} \frac{r'}{n} = t'$, $\lim\limits_{n \to \infty} \frac{r}{n} = t$ and $c_{10} \leq \min\{t', 1-t\}$,

then,

$$(1.5.4) \qquad \lim_{n \to \infty} \inf_{\Omega} P[CS|R_{10}] = \int_0^\infty H^{k-1}(\frac{x}{c_{10}}) dH(x)$$

where $H(x)$ is the c.d.f. of T defined in 1.5.2.

Proof. $P[CS|R_{10}] = P[T_{(k)} \geq c_{10}T_{(j)}, j=1, \ldots, k-1]$

where $T_{(j)} = -r' X_{(i)r',n} - X_{(i)r'+1,n} - \cdots - X_{(i)\ell_{(i)},n} + X_{(i)\ell_{(i)}+1,n} + \cdots$
$$+ (n-r+1)X_{(i)r,n}$$

and $X_{(i),1,n} \leq \cdots \leq X_{(i)\ell_{(i)},n} < 0 < X_{(i)\ell_{(i)}+1,n} < \cdots < X_{(i)n,n}$ are order statistics from $\pi_{(i)}$.

It is easy to see that

(1.5.5) $\quad \inf_{\Omega} P[CS|R_{10}] = P[T'_k \geq c_{10}T'_j, \; j=1,\ldots,k-1]$

where $T'_i = -r'Y_{i;r',n} - \cdots - Y_{i;\ell_{(i)},n} + Y_{i;\ell_{(i)}+1,n} + \ldots + (n-r+1)Y_{i;r,n}$

and $Y_{i;1,n} \leq \cdots \leq Y_{i;\ell_{(i)},n} < 0 < Y_{i;\ell_{(i)}+1,n} \leq \cdots \leq Y_{i,n,n}$ are order statistics from $F(x)$.

Let $\phi(x) = G^{-1}F(x)$.

(1.5.6) $\quad P[T'_k \geq c_{10}T'_j] = P[\phi(\frac{1}{c_{10}n}T'_k) \geq \phi(\frac{1}{n}T'_j), \; j=1,\ldots,k-1]$.

By Corollary 2.9 of Lawrence [48] with $i_0 = \ell_{(i)}$, $j_0 = \ell_{(i)}+1$, and

$$a_j = \begin{cases} 0 & 1 \leq j < r' \\ \dfrac{-r'}{n} & j = r' \\ -\dfrac{1}{n} & r' < j \leq \ell_{(i)} \\ \dfrac{1}{n} & \ell_{(i)} < j < r \\ \dfrac{n-r+1}{n} & j = r \\ 0 & n \geq j > r \end{cases} \quad ,$$

then

(1.5.7) $\quad \phi(\frac{1}{n}T'_i) \leq \frac{-r'}{n}\phi(Y_{i;r',n}) - \cdots - \frac{1}{n}\phi(Y_{i;\ell_{(i)},n})$

$\qquad\qquad + \frac{1}{n}\phi(Y_{i;\ell_{(i)}+1,n}) + \ldots + \frac{n-r+1}{n}\phi(Y_{i;r,n}), \; i=1,\ldots,k-1.$

Also by Corollary 2.10 of Lawrence [48] with $i_0 = r'-1$, $j_0 = n+1$,

$$
a_j = \begin{cases}
0 & 0 \le j < r' \\[2mm]
\dfrac{-r'}{nc_{10}} & j = r' \\[2mm]
\dfrac{-1}{nc_{10}} & r' < j \le \ell(k) \\[2mm]
\dfrac{1}{nc_{10}} & \ell(k) < j < r \\[2mm]
\dfrac{n-r+1}{nc_{10}} & j = r \\[2mm]
0 & j > r
\end{cases},
$$

$\dfrac{-r'}{nc_{10}} \le -1$ and $\dfrac{n-r+1}{nc_{10}} \ge 1$,

then

$$
(1.5.8) \quad \phi\left(\frac{1}{nc_{10}} T_k'\right) \ge \frac{-r'}{nc_{10}} \phi(Y_{k;r',n}) - \ldots - \frac{1}{nc_{10}} \phi(Y_{k;\ell(k),n})
$$

$$
+ \frac{1}{nc_{10}} \phi(Y_{k;\ell(k)+1,n}) + \ldots + \frac{n-r+1}{nc_{10}} \phi(Y_{k;r,n}).
$$

Now $\phi(Y_{j;i,n}) \underset{st}{=} V_{j;i,n}$, $j = 1,\ldots,k$, $i = 1,\ldots,n$, where $V_{j;i,n}$
is the i-th order statistic from $G(x)$, $j = 1,\ldots,k$.

Let

$$
T_i'' = -r' V_{i;r',n} - \ldots - V_{i;\ell(i),n} + V_{i;\ell(i)+1,n} + \ldots
$$

$$
+ (n-r+1) V_{i;r,n}, \quad i = 1,\ldots,k.
$$

From (1.5.6), (1.5.7) and (1.5.8), we obtain

$$
P[T_k' \ge c_{10} T_j', \; j \neq k] \ge P[T_k'' \ge c_{10} T_j'', \; j = 1,\ldots,k-1].
$$

Since G is symmetric about 0, as n becomes large,

$$
T_j'' \underset{st}{=} T, \quad j = 1,\ldots,k.
$$

Thus

$$P[CS|R_{10}] \geq \int_0^\infty H^{k-1}(\frac{x}{c_{10}})dH(x) \text{ if } c_{10} \leq \min\{\frac{r'}{n}, \frac{n-r+1}{n}\}.$$

This proves the theorem.

Corollary 1.5.1. If $F_i(x) = F(\frac{x}{\theta_i})$ $\forall x$ and $i = 1,\ldots,k$, $F \underset{s}{\prec} G$, F is symmetric about 0 and $g(x) = \frac{dG(x)}{dx} = \frac{1}{2} e^{-|x|}$, $-\infty < x < \infty$.

If $\lim_{n\to\infty} \frac{r'}{n} = t' < \frac{1}{2}$, $\lim_{n\to\infty} \frac{r}{n} = t > \frac{1}{2}$ and $c_{10} \leq \min\{t', 1-t\}$, then, as n is large,

$$(1.5.9) \quad \inf_{\Omega} P[CS|R_{10}] \approx \int_{-\infty}^\infty \phi^{k-1}[\frac{x}{c_{10}} + (\frac{1-c_{10}}{c_{10}})\sqrt{r-r'+1}\ ]d\phi(x)$$

Proof. Considering

$$T^* = \sum_{i=r'}^{\ell-1} -i(Y_{i,n} - Y_{i+1,n}) + \sum_{\ell+1}^r (n-i+1)(Y_{i,n} - Y_{i-1,n})$$

where $Y_{1,n} \leq \ldots \leq Y_{\ell,n} < 0 < Y_{\ell+1,n} < \ldots < Y_{n,n}$ are order statistics from $G(x)$.

Since $2 T^*$ is distributed as a χ^2 r.v. with $2(r-r'+1)$ d.f. (see Lawrence [47]) and T^* has same distribution as T when n is large, then $2 T$ is distributed as χ^2 with $2(r-r'+1)$ d.f. By Theorem 1.5.1,

$$P[CS|R_{10}] \geq P[V_k \geq c_{10} \max_{1\leq j\leq k} V_j]$$

where $V_1,\ldots,V_k$ are i.i.d. from χ^2 with $2(r-r'+1)$ d.f. Since $V_i \sim N(2(r-r'+1), 4(r-r'+1))$, then

$$P[V_k \geq c \max_{1\leq j\leq k} V_j] \approx \int_{-\infty}^\infty \phi^{k-1}(\frac{x}{c_{10}} + \frac{1-c_{10}}{c_{10}} \sqrt{r-r'+1})d\phi(x)$$

This proves the corollary.

Assume $X_{1,n} \leq \cdots \leq X_{k,n} \leq 0 \leq X_{k+1},n \leq \cdots \leq X_{n,n}$ are the order statistics from distribution G where $\frac{d}{dx} G(x) = g(x) = \frac{1}{2} e^{-|x|}$, $-\infty < x < \infty$.

Define

(1.5.10) $\qquad V_i = \dfrac{\sum\limits_{j=1}^{i-1} H_n^{-1}(\frac{j}{n})}{H_n^{-1}(\frac{i}{n})}$, $i = 1,\ldots,n$,

where $H_n^{-1}(\frac{i}{n}) = \sum\limits_{j=1}^{i} gG^{-1}(\frac{j-1}{n}) (X_{j,n} - X_{j-1,n})$.

Now we want to get the limiting distribution of V_n (as n is large).

Let $T_n = \sqrt{n}[\frac{1}{n} V_n - \frac{1}{2}]$. To obtain the asymptotic distribution of T_n, we use the following result. Let $X_{1,n} < \cdots < X_{n,n}$ be the order statistics from G and $S_n = \frac{1}{n} \sum\limits_{i=1}^{n} L(\frac{i}{n}) X_{i,n}$.

$$\sigma^2 = \sigma^2(G) = 2 \iint\limits_{s<t} L(G(s))L(G(t))G(s)[1-G(t)]dsdt.$$

where L satisfies the condition (ii) of following lemma.

Lemma 1.5.1. (Moore [53])

If $\sigma^2 < \infty$ and

(i) $\quad E|X| = \int_0^1 |G^{-1}(u)| du < \infty$

(ii) L is continuous on $[0,1]$ except for jump discontinuities at $a_1,\ldots,a_M$, and L' is continuous and of bounded variation on $[0,1] - \{a_1,\ldots,a_M\}$,

then

(1.5.12) $\qquad \mathcal{L} \{\sqrt{n} [S_n - \int\limits_{-\infty}^{\infty} xL(G(x))dG(x)]\} \to N(0,\sigma^2)$.

Let

$$(1.5.11) \qquad S_n = \frac{1}{n} \sum_{i=1}^{n-1} H_n^{-1}(\frac{i}{n}) - \frac{1}{2} H_n^{-1}(1).$$

Then using an argument similar to the one in Barlow and Doksum [3]

$$S_n = \frac{1}{n} \sum_{i=1}^{n} L(\frac{i}{n}) X_{i,n}$$

where $L(\frac{i}{n}) = g \ G^{-1}(\frac{i-1}{n}) - n(\frac{1}{2} - \frac{i+1}{n})[g \ G^{-1}(\frac{i}{n}) - g \ G^{-1}(\frac{i-1}{n})].$

Since

$$g^{-1}G(x) = \begin{cases} x & \text{for} \ \ 0 < x \le \frac{1}{2} \\ 1-x & \text{for} \ \ \frac{1}{2} < x \le 1 \end{cases},$$

then

$$L(\frac{i}{n}) = \begin{cases} 2(\frac{i}{n}) - \frac{1}{2} & \text{for} \ \ \frac{i}{n} \le \frac{1}{2} \\ \frac{3}{2} - 2(\frac{i}{n}) & \text{for} \ \ \frac{i}{n} > \frac{1}{2} \end{cases}.$$

Thus we can consider

$$(1.5.12) \qquad S_n = \frac{1}{n} \sum_{i=1}^{n} L(\frac{i}{n}) X_{i,n}$$

$$\text{where } L(x) = \begin{cases} 2x - \frac{1}{2} & \text{for} \ \ 0 < x \le \frac{1}{2} \\ \frac{3}{2} - 2x & \text{for} \ \ \frac{1}{2} < x \le 1 \end{cases}.$$

We can show that

$$\int_{-\infty}^{\infty} xL(G(x))dG(x) = 0.$$

It follows from Lemma 1.5.1,

$$\mathcal{L}(\sqrt{n} \ S_n) \to N(0, \sigma^2(G)).$$

where

$$(1.5.13) \qquad \sigma^2(G) = 2 \int\limits_{s<t} \int L(G(s))L(G(t))G(s)(1-G(t))dsdt.$$

$$\sigma^2(G) = 2 \int\limits_0^1 [\int\limits_0^t \frac{L(s)}{g\ G^{-1}(s)}\ sds]\ \frac{L(t)}{g\ G^{-1}(t)}\ (1-t)dt$$

$$= 2 \int\limits_0^{\frac{1}{2}} [\int\limits_0^t \frac{L(s)}{g\ G^{-1}(s)}\ sds]\ \frac{L(t)}{g\ G^{-1}(t)}\ (1-t)dt$$

$$+ 2 \int\limits_{\frac{1}{2}}^1 [\int\limits_0^{\frac{1}{2}} \frac{L(s)}{gG^{-1}(s)}\ sds + \int\limits_{\frac{1}{2}}^t \frac{L(s)}{gG^{-1}(s)}\ sds]\ \frac{L(t)(1-t)}{g\ G^{-1}(t)}\ dt$$

$$= 2 \cdot \frac{1}{48} + 2 \cdot \frac{5}{48} = \frac{1}{4}\ .$$

By Theorem 2.1 of Barlow and Van Zwet [11], $H_n^{-1}(1) \rightarrow G(X_{n,n})$ a.s.

Since $G(X_{n,n}) \rightarrow 1$, therefore $H_n^{-1}(1) \rightarrow 1$ a.s.

Now $T_n = \dfrac{\sqrt{n}S_n}{H_n^{-1}(1)}$, hence by Slutsky's theorem,

we have the following result.

Theorem 1.5.2.

$$(1.5.14) \qquad \mathcal{L}\ (\sqrt{n}\ (\frac{V_n}{n} - \frac{1}{2})) \rightarrow N(0, \frac{1}{4})$$

1.6. Selecting a subset which contains all populations better than
       a control.

In addition to $\pi_1, \ldots, \pi_k$, we have a control population π_0. Let π_i have a continuous distribution F_i, $i = 0,1,\ldots,k$. We are given a sample size n from each of the (k+1) populations π_i, $i = 0,1,\ldots,k$. Let $\xi_{\alpha i}$ be the unique α-quantile, $i = 0,1,\ldots,k$. A population π_i ($i = 1,2,\ldots,k$) is defined as better than the control π_0 if $\xi_{\alpha i} > \xi_{\alpha 0}$.

We want to select a subset such that the probability is at least P^* ($\frac{1}{k} < P^* < 1$) that all populations π_i ($i = 1,\ldots,k$) for which $\xi_{\alpha i} \geq \xi_{\alpha 0}$ will be included in the subset. We regard any such selection a correct selection (CS). We assume that for a fixed i, $i = 1,\ldots,k$, either $F_i(x) \geq F_0(x)$ or $F_i(x) \leq F_0(x)$ $\forall x > 0$. Thus π_i is better than π_0 iff $F_i(x) \leq F_0(x)$ $\forall x > 0$. Here π_0 is not known and we assume $F_0 \underset{*}{<} G$. Let T_i be the r-th order statistic from F_i where $r \leq (n+1)\alpha < r+1$ and let H_i be the c.d.f. of the r-th order statistic from F_i, $i = 0,1,\ldots,k$. We propose the rule R_{11} as follows,

R_{11}: Select π_i if and only if

$$(1.6.1) \qquad T_i \geq c_{11} T_0$$

where c_{11} ($0 < c_{11} < 1$) is determined to satisfy the basic probability requirement. Let t be the number of π_i's better than π_0 and let Δ be the index set for those π_i, t is unknown. Let $G_r(x)$ be the c.d.f. of the r-th order statistic of sample size r from G. Let Ω be the set of (k+1)-tuples $(F_0, F_1, \ldots, F_k)$.

Theorem 1.6.1. If $F_0 \underset{*}{<} G$ and for a fixed i, $i = 1,\ldots,k$, either $F_i(x) \geq F_0(x)$ or $F_i(x) \leq F_0(x)$, $x > 0$, then

$$(1.6.2) \quad \inf_{\Omega} P[CS|R_{11}] = \int_0^\infty \bar{G}_r^k(c_{11}x)dG_r(x) \text{ where } \bar{G}_r(x)=1-G_r(x).$$

Proof. $P[CS|R_{11}] = P[T_i \geq c_{11} T_0 \quad i \in \Delta]$

$$= \int_0^\infty \prod_{i\in\Delta} [1-H_i(c_{11}x)]dH_0(x)$$

Since $F_i(x) \leq F_0(x)$ for $i \in \Lambda$, then $H_i(x) \leq H_0(x)$.

Thus

$$P[CS|R_{11}] \geq \int_0^\infty \prod_{i \in \Lambda} [1-H_0(c_{11}x)]dH_0(x)$$

$$= \int_0^\infty [1-H_0(c_{11}x)]^t \, dH_0(x) \geq \int_0^\infty [1-H_0(c_{11}x)]^k dH_0(x)$$

$$= P[\min_{1 \leq i \leq k} Z_i \geq c_{11} Z_0]$$

where $Z_0, Z_1, \ldots, Z_k$ are i.i.d. with c.d.f. $H_0(x)$. Since $F_0 \overset{<}{*} G$ (see Barlow and Gupta [5]), then

$$P[\min_{1 \leq i \leq k} Z_i \geq c_{11} Z_0] \geq \int_0^\infty [\mathring{G}_r(c_{11}x)]^k \, dG_r(x).$$

This completes the proof.

1.7. Quantile selection rule for a restricted family of distributions

We assume that each $F_i(G)$ has a unique α-quantile $\xi_{i\alpha}(\eta_\alpha)$ and $F_i(0) = 0 = G(0)$, $i = 1,2,\ldots,k$.

Let $\xi_{[1]\alpha} \leq \cdots \leq \xi_{[k]\alpha}$ be ordered values of $\xi_{i\alpha}$, $i = 1,\ldots,k$. A best population is defined as the one having $\xi_{[k]\alpha}$. Let T_i denote the r-th order statistic from F_i, where $r \leq (n+1)\alpha < r+1$. The rule we propose is

R_{12}: Select π_i if and only if

$$(1.7.1) \qquad T_i \geq \max_{1 \leq j \leq k} T_j - c_{12}$$

where c_{12} ($0 < c_{12} \leq 1$) is determined so as to satisfy the probability requirement $P[CS|R_{12}] \geq P^*$.

As before let Ω be the set of all k-tuples $(F_1,..,F_k)$. Also if F has a unique α-quantile ξ_α, we define $F \underset{S*}{<} G$ if and only if $F(0) = 0 = G(0)$ and $G^{-1}F(x)$ is concave-convex about the ξ_α, on the support of F.

For example, if $G(x) = 1-e^{-x}$, then $F \underset{S*}{<} G$ is equivalent to saying that $F(x)$ is DFR for $x \leq \xi_\alpha$ and $F(x)$ is IFR for $x > \xi_\alpha$, on the support of F. We are interested in this "turning point" ξ_α. Let $H_{r,i}(x)$ be the c.d.f. of the r-th order statistic from $F_{[i]}$ and let $G_r(x)$ be the c.d.f. of the r-th order statistic from G.

<u>Theorem 1.7.1</u>. If $F_{[i]}(x) \geq F_{[k]}(x)$, $x \geq 0$ and $i = 1,...,k$, $F_{[k]} \underset{S*}{<} G$ and $F_{[k]}(y+\xi_{[k]\alpha}) \geq G(y+\eta_\alpha)$, for $y > 0$, then

(1.7.2) $$\underset{\Omega}{\text{Inf}} \, P[CS|R_{12}] = \int_0^\infty G_r^{k-1}(x+c_{12})dG_r(x)$$

where $G_r(x)$ is the cdf of r-th order statistic from G.

Proof.

(1.7.3) $$P[CS|R_{12}] = \int_0^\infty \prod_{i=1}^{k-1} H_{r,i}(x+c_{12})dH_{r,k}(x)$$

$$\geq \int_0^\infty [H_{r,k}(x+c_{12})]^{k-1} dH_{r,k}(x) = P[X_{r,k} \geq \underset{1\leq i \leq k-1}{\max} X_{r,i} - c_{12}]$$

where $X_{r1},...,X_{r,k}$ are i.i.d. r.v.'s with distribution $H_{r,k}$.

Let $\varsigma(x) = G^{-1}F_{[k]}(x) = G_r^{-1}H_{r,k}(x)$. Since $G^{-1}F_{[k]}(x)$ is concave-convex about $\xi_{[k]\alpha}$, then $\varphi'(x)$ is decreasing in x for $0 < x < \xi_{[k],\alpha}$

and $\varphi'(x)$ is increasing in x for $x > \xi_{[k],\alpha}$.

So that $\varphi'(x) \geq \varphi'(\xi_{[k],\alpha}) \ \forall \ x > 0$. Since $F_{[k]}(y+\xi_{[k]\alpha}) \geq G(y+\eta_\alpha)$

for $y > 0$, then $F_{[k]}(x) \geq G(x+\eta_\alpha-\xi_{[k],\alpha})$ for $x \geq \xi_{[k]\alpha}$.

Hence
$$\frac{G^{-1}F_{[k]}(x) - \eta_\alpha}{x-\xi_{[k]\alpha}} \geq 1.$$

Let $A(x) = G^{-1}F_{[k]}(x) - G^{-1}F(\xi_{[k]\alpha}) = G^{-1}F_{[k]}(x) - \eta_\alpha$, then

$$\frac{A(x) - A(\xi_{[k]\alpha})}{x-\xi_{[k]\alpha}} \geq 1.$$

Let $X \rightarrow \xi_{[k]\alpha}$, then $A'(\xi_{[k]\alpha}) \geq 1$.

Since
$$\frac{d}{dx} A(x) = A'(x) = \frac{f_{[k]}(x)}{g[G^{-1}F_{[k]}(x)]} \ ,$$

then
$$A'(\xi_{[k]\alpha}) = \frac{f_{[k]}(\xi_{[k]\alpha})}{g(\eta_\alpha)} \geq 1.$$

Hence $\varphi'(x) \geq 1 \quad \forall \ x > 0$.

Since $\dfrac{\varphi(\max X_{r,i}) - \varphi(X_{r,k})}{\max X_{r,i} - X_{r,k}} = \varphi'(t)$ for some t such that

$X_{r,k} \leq t \leq \max\limits_{1\leq i\leq k} X_{r,i}$ and $\varphi(X_{r,i}) = Y_{r,i}$ has distribution G_r,

then

$$\max\limits_{1\leq i\leq k} Y_{r,i} - Y_{r,k} \geq \max\limits_{1\leq i\leq k} X_{r,i} - X_{r,k}.$$

From (1.7.3), we obtain

$$P[CS|R_{12}] \geq P[\max\limits_{1\leq i\leq k} Y_{r,i} - Y_{r,k} \leq c_{12}]$$
$$= \int_0^\infty G_r^{k-1}(x+c_{12})dG_r(x).$$

This completes the proof.

1.8. Estimation of ordered parameters

We assume that $F_i(x) = F(\frac{x}{\theta_i})$ $\forall$ x and $i = 1,\ldots,k$, where F_i

and θ_i are unknown and $\theta_i > 0$, $i = 1,\ldots,k$. Let

$\theta_{[1]} \leq \cdots \leq \theta_{[k]}$ be ordered values of θ_i's. We assume that

each $F_i(F)$ has a unique α-quantile $\xi_i(\xi)$, $i = 1,\ldots,k$.

Let $\xi_{[1]} \leq \cdots \leq \xi_{[k]}$ be ordered values of ξ_i's. We are given a

sample of size n from each of the k populations. Let T_i be

the total life statistic until r-th failure from F_i, $i=1,\ldots,k$,

$T^*(T)$ be the total life statistic until r-th failure from $F(G)$

and let T_i^* be the r-th order statistic from F_i, $i = 1,\ldots,k$,

where $r \leq (n+1)\alpha < r+1$.

Let $T_{[1]} \leq \cdots \leq T_{[k]}$ be order values of T_i's

and let $T^*_{[1]} \leq \cdots \leq T^*_{[k]}$ be order values of T^*_i's. Let $T_{(i)}$ and

$T^*_{(i)}$ be associated with $\theta_{[i]}$, $i = 1,\ldots,k$.

(A) Estimation of $\theta_{[i]}$ based on $T_{[i]}$.

Theorem 1.8.1. If $F \underset{c}{<} G$, $F(0) = G(0) = 0$ and if

$\theta_0 = \int_0^\infty xdF(x) = \int_0^\infty xdG(x)$ where θ_0 is known, then

(1.8.1) $P[\theta_{[i]} \leq \frac{1}{\Delta} T_{[i]}] \geq \beta$ if $\Delta \leq \theta_0(n-r+1)$,

where $\Delta = W^{-1}(1 - \beta^{\frac{1}{k-i+1}})$ and $W(x)$ is the c.d.f. of T from G.

Proof. $P[\theta_{[i]} \leq \frac{1}{\Delta} T_{[i]}] = P[T_{[i]} \geq \Delta \theta_{[i]}]$.

Since $P[T_{[i]} \leq t]$ is a non-increasing function of θ_ℓ $(1 \leq \ell \leq k)$ (see Chen and Dudewicz [20]), then

$$P[T_{[i]} \geq \Delta \; \theta_{[i]}] \geq P^{k-i+1}[T_{(i)} \geq \Delta \; \theta_{[i]}]$$

$$= P^{k-i+1}[T^\star \geq \Delta].$$

By Theorem 4.6 of Barlow and Proschan [8],

$$P^{k-i+1}[T^\star \geq \Delta] \geq P^{k-i+1}[T \geq \Delta]$$

$$= \{1-W(\Delta)\}^{k-i+1} = \{1-(1-\beta^{\frac{1}{k-i+1}})\}^{k-i+1} = \beta.$$

Let $\chi_\alpha^2(2r)$ denote the 100α percent point of a chi-square distribution with 2r d.f. Again, by Theorems 3.3 and 2.4 of Barlow and Proschan [9], we have the following theorems.

<u>Theorem 1.8.2.</u> If F is IFRA and $\int_0^\infty xdF(x) = \theta_0$ where θ_0 is known, then

(1.8.2) $P[\theta_{[i]} \leq c_1 T_{[i]}] \geq \beta_1$ where

$$c_1 = \begin{cases} \dfrac{2}{\theta_0 \chi_{1-\beta'}^2(2r)} & \text{if } \chi_{1-\beta'}^2(2r) \leq 2(n-r+1) \\[2ex] \dfrac{1}{(n-r+1)\theta_0} & \text{if } \chi_{1-\beta'}^2(2r) \geq 2(n-r+1) \end{cases}$$

$$\text{where } \beta' = \beta_1^{\frac{1}{k-i+1}}$$

<u>Theorem 1.8.3.</u> If F is IFR and $\int_0^\infty xdF(x) = \theta_0$ where θ_0 is known, then

(1.8.3) $P[\theta_{[i]} \geq c_2 T_{[i]}] \geq \beta_2$ where $c_2 = \dfrac{2nk}{\theta_0 x^2_{\beta''}(2r)}$

and $\beta'' = \beta_2^{\frac{1}{i}}$.

Let β_1, β_2 $(0 < \beta_1, \beta_2 < 1)$ be such that $\beta_1 + \beta_2 - 1 = \gamma$ where γ is fixed and $0 < \gamma < 1$. By Theorems 1.8.2, and 1.8.3 and the fact that $P(AB) \geq P(A) + P(B) - 1$ with $A = \{\theta_{[i]} \leq c_1 T_{[i]}\}$ and $B = \{\theta_{[i]} \geq c_2 T_{[i]}\}$, we have the following theorem for two-sided confidence interval of $\theta_{[i]}$.

<u>Theorem 1.8.4</u>. If F is IFR and $\int_0^\infty x dF(x) = \theta_0$ where θ_0 is known, then

(1.8.4) $P[c_2 T_{[i]} \leq \theta_{[i]} \leq c_1 T_{[i]}] \geq \gamma$

where c_1, c_2 are defined as above.

(B) Estimation of $\xi_{[i]}$ based on $T_{[i]}$

<u>Theorem 1.8.5</u>. If $F \underset{*}{<} G$, then

(1.8.6) $P[\xi_{[i]} \leq c_1 T_{[i]}] \geq \beta_1$ where $c_1 = \begin{cases} \Delta_1 & \text{if } \Delta_1(n-r+1) \geq 1 \\[2ex] \dfrac{1}{n-i+1} & \text{if } \Delta_1(n-r+1) \leq 1 \end{cases}$

where $\Delta_1 = \dfrac{G^{-1}(\alpha)}{W^{-1}(1-\beta_1^{\frac{1}{k-i+1}})}$ and

W(x) is the c.d.f. of T from G.

Proof. $P[\xi_{[i]} \leq c_1 T_{[i]}]$

$$\geq P^{k-i+1}[T_{(i)} \geq \frac{1}{c_1} \xi_{[i]}] \qquad \text{(see Theorem 1.8.1)}$$

$$= P^{k-i+1}[T_{(i)} \geq \frac{1}{c_1} \cdot \xi_{[i]}]$$

$$= P^{k-i+1}[T^\star \geq \frac{1}{c_1} \xi]$$

$$= P^{k-i+1}[T^\star \geq \frac{1}{c_1} F^{-1}(\alpha)] = P^{k-i+1}[F(c_1 T^\star) \geq \alpha].$$

In a manner similar to the proof of Theorem 3.1 of Barlow and

Proschan [9], we can show that

$$P[F(c_1 T^\star) \geq \alpha] \geq \beta_1^{\frac{1}{k-i+1}}.$$

Hence

$$P[\xi_{[i]} \leq c_1 T_{[i]}] \geq \beta_1.$$

Looking at Theorem 2.1 of Barlow and Proschan [9], we can

have the following theorem.

Theorem 1.8.6. If $F \underset{c}{\prec} G$ and $F(0) = 0 = G(0)$, then

(1.8.7) $P[\xi_{[i]} \geq c_2 T_{[i]}] \geq \beta_2$ where $c_2 = \begin{cases} \Delta_2 & \text{if } \Delta_2 n \leq 1 \\ \\ \frac{1}{n} & \text{if } \Delta_2 n \geq 1 \end{cases}$,

$$\Delta_2 = \frac{G^{-1}(\alpha)}{W^{-1}(\frac{1}{2})} \text{ and}$$

$W(x)$ is the c.d.f. of T from G.

Let β_1, β_2 $(0 < \beta_1, \beta_2 < 1)$ be such that $\beta_1 + \beta_2 - 1 = \gamma$, where γ is

fixed and $0 < \gamma < 1$. By Theorem 1.8.5 and Theorem 1.8.6, and the fact that

if $F(0) = 0 = G^{-1}(0) = 0$, then $F \underset{c}{\prec} G \Rightarrow F \underset{\star}{\prec} G$ we

have the following theorem for two-sided confidence interval of $\xi_{[i]}$.

<u>Theorem 1.8.7.</u> If $F \underset{c}{\prec} G$ and $F(0) = 0 = G^{-1}(0)$, then

(1.8.8) $P[c_2 T_{[i]} \leq \xi_{[i]} \leq c_1 T_{[i]}] \geq \gamma$

(c) Estimation of $\xi_{[i]}$ based on $T^*_{[i]}$.

<u>Theorem 1.8.8.</u> If $F \underset{*}{\prec} G$, then

(1.8.9) (1) $P[\xi_{[i]} \leq \dfrac{T^*_{[i]}}{\Delta_1}] \geq \beta$ if $\Delta_1 = \dfrac{G_j^{-1}(1-\beta^{\frac{1}{k-i+1}})}{G^{-1}(\alpha)} \leq 1$

    (2) $P[\xi_{[i]} \geq \dfrac{T^*_{[i]}}{\Delta_2}] \geq \beta$ if $\Delta_2 = \dfrac{G_j^{-1}(\beta^{\frac{1}{1}})}{G^{-1}(\alpha)} \geq 1.$

Proof. First, we want to prove (1).

$P[\xi_{[i]} \leq \dfrac{T^*_{[i]}}{\Delta_1}] = P[T^*_{[i]} \geq \Delta_1 \xi_{[i]}]$

$\geq P^{k-i+1}[T^*_{(i)} \geq \Delta_1 \xi_{[i]}]$ (see Theorem 1.8.5)

$= \{1-P[T^*_{(i)} \leq \Delta_1 \xi_{[i]}]\}^{k-i+1}$

$= \{1-F_j(\Delta_1 \xi_{[i]} \cdot \dfrac{\xi}{\xi_{[i]}})\}^{k-i+1} = \{1-F_j(\Delta_1 \xi)\}^{k-i+1}$

where $F_j(x)$ is the c.d.f. of the j-th order statistic from F.
Since $G^{-1}F(x) = G_j^{-1}F_j(x)$, it follows that $F_j(x) = G_j G^{-1}F(x)$. Hence

$$F_j(\Delta_1 \xi) = G_j G^{-1} F(\Delta_1 \xi)$$

$$\leq G_j[\Delta_1 G^{-1} F(\xi)] \qquad \text{(since } F \underset{*}{\prec} G \text{ and } 0 < \Delta_1 < 1)$$

$$= G_j[\Delta_1 G^{-1}(\iota)]$$

$$= G_j[G_j^{-1}(1-e^{\frac{1}{k-i+1}})]$$

$$= 1-e^{\frac{1}{k-i+1}}$$

so that

$$(1-F_j(\Delta_1 \xi))^{k-i+1} \leq (1-(1-e^{\frac{1}{k-i+1}}))^{k-i+1} = \beta.$$

Now, we want to prove (2).

$$P[\xi_{[i]} \geq \frac{T^*_{[i]}}{\Delta_2}] = P[T^*_{[i]} \leq \Delta_2 \xi_{[i]}]$$

$$\geq P^i[T^*_{(i)} \leq \Delta_2 \xi_{[i]}]$$

$$= F_j^i(\Delta_2 \xi)$$

$$= \{G_j G^{-1} F(\Delta_2 \xi)\}^i$$

$$\geq G_j[\Delta_2 G^{-1} F(\xi)]^i \qquad \text{(since } F \underset{*}{\prec} G \text{ and } \Delta_2 \geq 1)$$

$$= \{G_j(\Delta_2 G^{-1}(\alpha))\}^i = \{G_j G_j^{-1}(\alpha^{\frac{1}{i}})\}^i = \beta.$$

This completes the proof.

CHAPTER II

INTERVAL ESTIMATION AND SELECTION PROCEDURES

FOR A SET OF GOOD POPULATIONS

2.1. Introduction

In practice, one is always faced with the problem of selecting the better ones from a group of populations. Here, we consider such kind of problems. Suppose that we are given $k(k \geq 2)$ populations $\pi_1, \ldots, \pi_k$ with distributions $F(x, \theta_i)$ where θ_i lies in an interval Λ on the real line, $i = 1, \ldots, k$. The quality of the i-th population is characterized by the real-valued parameter θ_i. The population with the largest θ-value is called the best population. A population is considered a good one if its quality does not fall too much below that of the best population. If $d(\theta_i, \theta_j)$ is a suitable distance measure between θ_i and θ_j and if $\theta_{[k]} = \max(\theta_1, \ldots, \theta_k)$, population π_i is

$$\text{good,} \quad \text{if} \quad d(\theta_{[k]}, \theta_i) \leq \Delta$$

(2.1.1)

$$\text{bad,} \quad \text{if} \quad d(\theta_{[k]}, \theta_i) > \Delta$$

where Δ is a given positive constant. Let $\underline{X}_i = (X_{i1}, \ldots, X_{in})$, $i = 1, \ldots, k$ be mutually independent random samples, each of size n, from population π_i, $i = 1, \ldots, k$ respectively. Let $T_i = T(\underline{X}_i)$ be

an appropriate estimator of θ_i. Let $g_n(t,\theta_i)$ and $G_n(t,\theta_i)$ be the density function and the distribution function of T_i, respectively. The results of this chapter relate to two cases,

(i) $G_n(t,\theta_i) = G_n(t-\theta_i)$, $-\infty < t < \infty$, $-\infty < \theta_i < \infty$, and

(2.1.2)

(ii) $G_n(t,\theta_i) = G_n(\frac{t}{\theta_i})$, $t > 0$, $\theta_i > 0$.

If the distribution function $F(x,\theta_i)$ of π_i belongs to a location parameter family, then $F(x,\theta_i) = F(x-\theta_i)$. The distribution function $G_n(t,\theta_i)$ of T_i which is based on a random sample size n is not necessary of the form $G_n(t-\theta_i)$, but we can always find a statistic T_i such that T_i has the distribution function of the form $G_n(t-\theta_i)$. For example, let $\pi_1,\ldots,\pi_k$ be k normal populations with unknown means $\theta_1,\ldots,\theta_k$ and a common known variance σ^2. Let $T_i = \bar{X}_i$ be the sample mean of π_i based on a sample of size n. In this case, $G_n(x,\theta_i)$ (the c.d.f. of $\bar{X}_i$) is of the form $G_n(x,\theta_i) = G_n(x-\theta_i)$, where $G_n(x) = \Phi(\frac{\sqrt{n}}{\sigma} x)$.

If the distribution function $F(x,\theta_i)$ of π_i belongs to a scale parameter family, then $F(x,\theta_i) = F(\frac{x}{\theta_i})$, $x > 0$, $\theta_i > 0$. As before we can find a statistic T_i such that T_i has the distribution function of the form $G_n(\frac{t}{\theta_i})$. The results of this chapter deal with the cases when $G_n(t,\theta_i)$ is one of the two forms in (2.1.2).

In Definition (2.1.1) we take d as d_L or d_S, respectively, whenever θ_i is a location parameter or is a scale parameter for the density of T_i. d_L and d_S are defined as $d_L(a,b) = a-b$, $d_S(a,b) = \frac{a}{b}$.

Estimation of $\theta_{[k]}$ based on $(T_1, \ldots, T_k)$ has been considered by several authors. Construction of two-sided and one-sided confidence intervals for $\theta_{[k]}$ based on

(2.1.1) $$T_{[k]} = \max_{1 \leq i \leq k} T_i$$

was considered by Dudewicz [24], Dudewicz and Tong [26], Saxena and Tong [65] and others.

For problem of selecting the good populations from a given collection was considered by Desu [21], Carroll, Gupta and Huang [19] and others.

We consider confidence interval I which is based on $T_{[k]}$. In this chapter, we are interested in finding the smallest sample size N such that the probability that I contains at least one good population is at least P* where P* is a specified number, $0 < P* < 1$. The probability that I contains all good populations and the probability that I excludes all the bad population are discussed.

In Section 2.2, we investigate the above problems for location parameter case. The infima of coverage probabilities are obtained. Some special cases are discussed. In Section 2.3, we investigate the above problems for scale parameter case. In Section 4, we discuss the above problem for the means of normal populations and for the scale parameters of the gamma populations.

2.2. Results for the location parameter case.

In this section, we assume that θ_i is a location parameter for T_i, $i = 1, \ldots, k$. It is assumed that there is no a priori

knowledge about the θ_i's. For arbitrary fixed d_1, $d_2 \ni d_1+d_2 > 0$, the confidence interval is defined as

(2.2.1) $I_1 = (T_{[k]}-d_1, \; T_{[k]} + d_2)$

where $T_{[k]} = \max\limits_{1 \leq i \leq k} T_i$.

For the location parameter case, we say that population π_i is

(2.2.2)   good   if $\theta_i \geq \theta_{[k]}-\Delta,$

                   bad   if $\theta_i < \theta_{[k]}-\Delta,$

where $\theta_{[k]} = \max\limits_{1 \leq i \leq k} \theta_i$.

For given $\underline{\theta} = (\theta_1,\ldots,\theta_k)$ and $\Delta > 0$, let $\alpha_{\underline{\theta}}(1)$ denote the probability that I_1 contains at least one good population, let $\alpha_{\underline{\theta}}(2)$ denote the probability that I_1 contains all good populations and let $\alpha_{\underline{\theta}}(3)$ denote the probability that I_1 excludes all the bad populations.

Let us denote the ordered θ-values by $\theta_{[1]} \leq \cdots \leq \theta_{[k]}$ and let Ω be the parameter space which is the collection of all possible parameter vectors $\underline{\theta} = (\theta_1,\ldots,\theta_k)$. For given $\Delta > 0$, let m denote the unknown number of bad populations in the given collection of k populations. Clearly we have $0 \leq m \leq k-1$.

Let

(2.2.3) $\Omega_m = \{\underline{\theta}: \theta_{[1]} \leq \cdots \leq \theta_{[m]} < \theta_{[k]}-\Delta \leq \theta_{[m+1]} \leq \cdots \leq \theta_{[k]}\}.$

Then

(2.2.4) $\Omega = \bigcup\limits_{m=0}^{k-1} \Omega_m.$

We need the following lemma before we prove some theorems.

Lemma 2.2.1. Define $f(r) = AC^r - BD^r$, $r = 0,1,\ldots,m$ where $A \geq B > 0$. If either (i) $1 < C \leq D$ or (ii) $1 > C \geq D > 0$, then

(2.2.4) $\qquad \min_{0 \leq r \leq m} f(r) = \min[f(0), f(m)]$.

Proof. Assume $1 < C \leq D$. For $r = 0,1,\ldots,m-1$,

$\qquad f(r+1) - f(r) = AC^r(C-1) - BD^r(D-1)$

$\qquad f(r+1) \leq f(r)$ if and only if $\frac{A}{B}(\frac{C}{D})^r \leq \frac{D-1}{C-1}$, $r = 0,1,\ldots,m-1$.

Since $\qquad \frac{C}{D} \leq 1 \Rightarrow \frac{A}{B}(\frac{C}{D})^{r+1} \leq \frac{D-1}{C-1}$.

hence

$\qquad\qquad f(r+2) \leq f(r+1)$.

Similarly, one can show that if $f(r+1) \geq f(r)$ then $f(r) \geq f(r-1)$. Hence the lemma follows.

If $1 > C \geq D > 0$, then by a similar argument as above, the result follows.

This completes the proof of the lemma.

Now we want to discuss the probability that I_1 contains at least one good population and the infimum of this probability over Ω. We need the following lemma.

Lemma 2.2.2. If the family of density functions $\{g_n(t-\theta): \theta \in \Lambda_1\}$ has a monotone likelihood ratio, for every $n \geq 1$, then for arbitrary fixed d_1, d_2 satisfying $d_1 + d_2 > \Lambda$,

(2.2.5) $\quad \inf_{\Omega_m} \alpha_\Omega(1) = \min\{G_n^m(\Lambda+d_1)G_n^{k-m}(d_1) - G_n^m(\Lambda-d_2)G_n^{k-1}(-d_2)$,

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad G_n^{k-m}(d_1) - G_n^{k-m}(-d_2)\}$.

Proof. For $\underline{\theta} \subset \underline{\Omega}_m$,

$$(2.2.6) \quad \alpha_{\underline{\theta}}(1) = P\{T_{[k]}-d_1 < \theta_{[i]} < T_{[k]}+d_2, \text{ for some } i, \ i=m+1,\ldots,k\}$$

$$= 1-P\{\theta_{[i]} \not\subset I, \text{ for each } i = m+1,\ldots,k\}$$

$$= 1-P\{T_{[k]}+d_2 \leq \theta_{[m+1]}\}-P\{T_{[k]}-d_1 \geq \theta_{[k]}\} \ \text{(since}$$

$$d_1+d_2 > \Delta)$$

$$= P\{T_{[k]}-d_1 < \theta_{[k]}\} - P\{T_{[k]}+d_2 \leq \theta_{[m+1]}\}$$

$$= \prod_{j=1}^{k} G_n[\theta_{[k]}+d_1-\theta_{[j]}] - \prod_{j=1}^{k} G_n[\theta_{[m+1]}-d_2-\theta_{[j]}]$$

Let $\delta_i = \theta_{[k]}-\theta_{[i]}$, $i = 1,\ldots,k$, then $0 = \delta_k < \delta_{k-1} < \ldots < \delta_{m+1} <$
$\Delta < \delta_m < \ldots < \delta_1$.

Hence

$$(2.2.7) \quad \alpha_{\underline{\theta}}(1) = \prod_{j=1}^{k} G_n(\delta_j+d_1) - \prod_{j=1}^{k} G_n(\delta_j-\delta_{m+1}-d_2).$$

It is easy to see that $\alpha_{\underline{\theta}}(1)$ is nondecreasing in δ_{m+1}, where
$\delta_{m+2} < \delta_{m+1} < \Delta$. Let $\delta_{m+1} = \delta_{m+2}$, then

$$\alpha_{\underline{\theta}}(1) = \prod_{\substack{j\neq m+1 \\ j\neq m+2}} G_n(\delta_j+d_1)G^2(\delta_{m+2}+d_1) - \prod_{\substack{j\neq m+1 \\ j\neq m+2}} G_n(\delta_j-\delta_{m+2}-d_2)G_n^2(-d_2).$$

In this case, also we can see that $\alpha_{\underline{\theta}}(1)$ is nondecreasing in δ_{m+2},

where $\delta_{m+3} < \delta_{m+2} < \Delta$. Let $\delta_{m+2} = \delta_{m+3}$. Repeating the process with
each δ_j, $j = m+1,\ldots,k-1$, thus we have

$$(2.2.8) \quad Q = \inf_{H_1} \alpha_{\underline{\theta}}(1) = \prod_{j=1}^{m} G_n(\delta_j+d_1)G_n^{k-m}(d_1) - \prod_{j=1}^{m} G_n(\delta_j-d_2)G_n^{k-m}(-d_2)$$

where $H_1 = \{(\delta_{m+1},\ldots,\delta_{k-1}): \ 0 < \delta_{k-1} < \ldots < \delta_{m+1} < \Delta\}$.

For $i = 1, 2, \ldots, m$,

$$\frac{\partial Q}{\partial \delta_i} = g(\delta_i + d_1)\{A - B\frac{g(\delta_i - d_2)}{g(\delta_i + d_1)}\}$$

where $A = G_n^{k-m}(d_1) \prod_{\substack{j=1 \\ j \neq i}}^{m} G_n(\delta_j + d_1)$ and $B = G_n^{k-m}(-d_2) \prod_{\substack{j=1 \\ j \neq i}}^{m} G_n(\delta_j - d_2)$.

Since $A > B$ and $\frac{g(\delta_i - d_2)}{g(\delta_i + d_1)}$ is non-increasing in $\delta_i > \Delta$, $i = 1, \ldots, m$,

then Q is greater than or equal to the right hand side of equation

(2.2.8) with either $\delta_i = \Delta$ or $\delta_i = \infty$, $i = 1, \ldots, m$.

Tnus

(2.2.9) $Q \geq \min_{0 \leq r \leq m} \{G_n^r(\Delta + d_1)G_n^{k-m}(d_1) - G_n^r(\Delta - d_2)G_n^{k-m}(-d_2)\}.$

By Lemma 2.2.1, we get

(2.2.10) $\inf_{\Omega_m} \alpha_{\underline{\theta}}(1) = \min\{G_n^{k-m}(d_1) - G^{k-m}(-d_2), G_n^m(\Delta + d_1)G^{k-m}(d_1) - $
$$G_n^m(\Delta - d_2)G^{k-m}(-d_2)\}.$$

This completes the proof.

<u>Theorem 2.2.1.</u> If the family of density functions $\{g_n(t-\theta): \theta \in \Lambda_1\}$

has montone likelihood ratio, for every $n \geq 1$, then for arbitrary

fixed d_1, d_2 satisfying $d_1 + d_2 > \Delta$,

(2.2.11) $\inf_{\Omega} \alpha_{\underline{\theta}}(1) = \min\{G_n(d_1) - G_n(-d_2), G_n^k(d_1) - G_n^k(-d_2),$
$$G_n^{k-m}(d_1)G_n^m(\Delta + d_1) - G_n^{k-m}(-d_2)G^m(\Delta - d_2), m = 1, \ldots, k-1\}.$$

Proof. Since $\Omega = \bigcup_{m=0}^{k-1} \Omega_m$, by Lemma 2.2.2, we have

(2.2.12) $\inf_{\Omega} \alpha_{\underline{\theta}}(1) = \min_{0 \leq m \leq k-1} \{\min[A^{k-m}B^{k} - B^{k-B}, A^{k-m}C^m - B^{k-m}D^m]\}$

where $A = G_n(d_1)$, $B = G_n(-d_2)$, $C = G_n(\wedge + d_1)$ and $D = G_n(\wedge - d_2)$.

By Lemma 2.2.1, $\min\limits_{0 \leq m \leq k-1} [A^{k-m} - B^{k-m}] = \min[A-B, A^k - B^k]$. From

(2.2.12), we get

$$(2.2.13) \quad \inf \iota_\theta(1) = \min\{A-B, A^k - B^k, A^{k-m}C^m - B^{k-m}D^m, m=1,2,\ldots,k-1\}.$$

This completes the proof.

<u>Corollary 2.2.1.</u> Let the family of density functions $\{g_n(t-\theta):$ $\cdot \in \theta_1\}$ have a montone likelihood ratio, for every $n \geq 1$. For arbitrary fixed d_1, d_2 satisfying $g(-d_2) \geq g(d_1)$ and $d_1 + d_2 > \varepsilon$, then

$$(2.2.14) \quad \inf \iota_-(1) = \min\{G_n(d_1) - G_n(-d_2), G_n^k(d_1) - G_n^k(-d_2),$$
$$G_n(d_1)G_n^{k-1}(\wedge + d_1) - G_n(-d_2)G_n^{k-1}(\cdot - d_2)\}.$$

Proof. By Theorem 2.2.1, we get

$$(2.2.15) \quad \inf \iota_\theta(1) = \min\{A-B, A^k - B^k, A^{k-m}C^m - B^{k-m}D^m, m=1,\ldots,k-1\}$$

where $A = G_n(d_1)$, $B = G_n(-d_2)$, $C = G_n(\wedge + d_1)$ and $D = G_n(\wedge - d_2)$.
Consider

$$(2.2.16) \quad f(m) = A^{k-m}C^m - B^{k-m}D^m$$
$$= A^k \left(\frac{C}{A}\right)^m - B^k \left(\frac{D}{B}\right)^m, \quad m = 0,1,\ldots,k-1.$$

Let

$$\psi(x) = \frac{G_n(x+d_1)}{G_n(x-d_2)}$$

$$\frac{d\psi(x)}{dx} = \frac{g_n(x+d_1)}{G_n^2(x-d_2)} \left\{ G_n(x-d_2) - G_n(x+d_1) \frac{g_n(x-d_2)}{g_n(x+d_1)} \right\}$$

since $d_2 \geq -d_1$, $\dfrac{g(x-d_2)}{g(x-d_1)}$ is nondecreasing in $x > 0$ and $g(-d_2) \geq g(d_1)$,

then $\dfrac{d\psi(x)}{dx} \leq 0$ for $x > 0$. Thus $\psi(\Delta) = \dfrac{C}{D} \leq \psi(0) = \dfrac{A}{B}$.

By Lemma 2.2.1 and $1 < \dfrac{C}{A} \leq \dfrac{D}{B}$, then

$$(2.2.17) \qquad \min_{0 \leq m \leq k-1} f(m) = \min\{f(0), f(k-1)\}.$$

Hence from (2.2.15) and (2.2.17),

$$\inf_{\underset{\sim}{\Omega}} \alpha_{\underset{\sim}{\theta}}(1) = \min\{A-B, A^k-B^k, f(0), f(k-1)\}$$

$$= \min\{A-B, A^k-B^k, AC^{k-1}-BD^{k-1}\}.$$

This completes the proof of the corollary.

Now we wish to discuss the probability that I_1 contains all good populations and the infimum of this probability over $\cdot\cdot$.

__Lemma 2.2.3.__ If the family of density functions $\{g_n(t-\cdot):$
$\cdot \in \Omega_1\}$ has a monotone likelihood ratio, for every $n \geq 1$, then for arbitrary fixed d_1, d_2 satisfying $d_1+d_2 > \Delta$,

$$(2.2.18) \quad \inf_{\underset{\sim}{\Omega}_m} \alpha_{\underset{\sim}{\theta}}(2) = \min_{0 \leq r \leq k-m-2} \min_{0 \leq \ell \leq m} \{[G_n(d_1)]^{k+\ell-m-r-1} G_n^{r+1}(-\Delta+d_1)$$

$$- [G_n(\Delta-d_2)]^{k+\ell-m-r-1} G_n^{r+1}(-d_2)\}.$$

Proof. For $\theta \in \Omega_m$,

$$(2.2.19) \quad \alpha_{\underset{\sim}{\theta}}(2) = P\{\theta_{[i]} \in [T_{[k]}-d_1, T_{[k]}+d_2], \ i = m+1,\ldots,k\}$$

$$= P\{\theta_{[m+1]} \geq T_{[k]}-d_1, \ \theta_{[k]} \leq T_{[k]}+d_2\}$$

$$= \prod_{j=1}^{k} G_n(\theta_{[m+1]}-\theta_{[j]}+d_1) - \prod_{j=1}^{k} G_n(\theta_{[k]}-\theta_{[j]}-d_2)$$

$$= \prod_{j=1}^{k} G_n(\lambda_j-\lambda_{m+1}+d_1) - \prod_{j=1}^{k} G_n(\lambda_j-d_2)$$

where $\delta_j = \delta_{[k]} - \delta_{[j]}$, $j = 1,\ldots,k$ and

$$0 = \delta_k < \delta_{k-1} < \cdots < \delta_{m+1} < \Lambda < \delta_m \cdots < \delta_1.$$

Since $\alpha_0(2)$ is nonincreasing in δ_{m+1}, where $\delta_{m+2} < \delta_{m+1} < \Lambda$, then $\alpha_0(2) \geq Q$ where

$$(2.2.20) \qquad Q = A \sum_{j=m+2}^{k-1} G_n(\delta_j - \Lambda + d_1) - B \sum_{j=m+2}^{k-1} G_n(\delta_j - d_2),$$

$$A = \prod_{j=1}^{m} G_n(\delta_j - \Lambda + d_1) G_n(d_1) G_n(-\Lambda + d_1) \text{ and}$$

$$B = \prod_{j=1}^{m} G_n(\delta_j - d_2) G_n(\Lambda - d_2) G_n(-d_2).$$

For $i = m+2,\ldots,k-1$,

$$\frac{\partial Q}{\partial \delta_i} = g_n(\delta_i - \Lambda + d_1)[A \sum_{\substack{j=m+2 \\ j \neq i}}^{k-1} G_n(\delta_j - \Lambda + d_1) - B \sum_{\substack{j=m+2 \\ j \neq i}}^{k-1} G_n(\delta_j - d_2) \frac{g_n(\delta_i - d_2)}{g_n(\delta_i - \Lambda + d_1)}].$$

Again, since $d_2 \geq \Lambda - d_1$ and $\frac{g_n(\delta_i - d_2)}{g_n(\delta_i - \Lambda + d_1)}$ is nondecreasing in δ_i,

$i = m+2,\ldots,k-1$, then Q is greater than or equal to the right hand

side of equation (2.2.20) with either $\delta_i = 0$ or $\delta_i = \Lambda$, $i = m+2,$

$\ldots, k-1$.

Let $H_1 = \{(\delta_{m+1}, \delta_{m+2}, \ldots, \delta_{k-1}) : 0 < \delta_{k-1} < \cdots < \delta_{m+1} < \Lambda\}$.

Hence,

$$(2.2.21) \quad \inf_{H_1} \alpha_0(2) = \min_{0 \leq r \leq k-m-2} [A G_n^r(-\Lambda+d_1) G_n^{k-m-2-r}(d_1) - B G_n^r(-d_2)$$
$$G_n^{k-m-2-r}(\Lambda - d_2)].$$

Let $H_2 = \{(\delta_1, \ldots, \delta_m) : \Lambda < \delta_m < \cdots < \delta_1\}$.

In a manner similar as above, for $r = 0,1,\ldots,k-m-2$, we can show

that

(2.2.22) $\quad \inf_{H_2}[AG_n^r(-\Lambda+d_1)G_n^{k-m-2-r}(d_1)-BG_n^r(-d_2)G_n^{k-m-2-r}(\Lambda-d_2)]$

$$= \min_{0\leq\ell\leq m} \{G_n^{r+1}(-\Lambda+d_1)[G_n(d_1)]^{k-m-1-r+\ell}-[G_n(\Delta-d_2)]^{k-m-1-r+\ell}$$
$$G_n^{r+1}(-d_2)\}.$$

From (2.2.21) and (2.2.22), we have our lemma.

Since $\Omega = \bigcup_{m=0}^{k-1} \Omega_m$, by Lemma 2.2.3, we can have the following theorem.

<u>Theorem 2.2.2.</u> If the family of density functions $\{g_n(t-\theta): \theta \in \Lambda_1\}$ has a monotone likelihood ratio, for every $n \geq 1$, then for arbitrary fixed d_1, d_2 satisfying $d_1+d_2 > \Delta$,

(2.2.23) $\quad \inf_{\Omega} \alpha_0(2) = \min_{0\leq m\leq k-1} \min_{0\leq r\leq k-m-2} \min_{0\leq\ell\leq m} \{[G_n(d_1)]^{k+\ell-m-r-1}$
$$G_n^{r+1}(-\Lambda+d_1)-[G_n(\Lambda-d_2)]^{k+\ell-m-r-1}G_n^{r+1}(-d_2)\}.$$

<u>Lemma 2.2.4.</u> Suppose that (i) $g_n(t) = g_n(-t) > 0$ for all t and (ii) the family of density functions $\{g_n(t-\theta): \theta \in \Lambda_1\}$ has a monotone likelihood ratio, for every $n \geq 1$. For arbitrary fixed d_1, d_2 satisfying $d_1 = d_2 = d > \Lambda$,

(2.2.24) $\quad \inf_{\Omega_m} \alpha_0(2) = \min\{A^{k-m-1}B-C^{k-m-1}D, \; A^{k-1}B-C^{k-1}D, \; AB^{k-m-1}-$

$$CD^{k-m-1}, \; A^{m+1}B^{k-m-1}-C^{m+1}D^{k-m-1}\}$$

where $A = G_n(d_1)$, $B = G_n(-\Lambda+d_1)$, $C = G_n(\Lambda-d_2)$ and $D = G_n(-d_2)$.

Proof. By Lemma 2.2.3, we have

(2.2.25) $\quad \inf_{\Omega_m} \alpha_0(2) = \min_{0\leq r\leq k-m-2} \min_{0\leq\ell\leq m} \{A^{k+\ell-m-r-1}B^{r+1}-C^{k+\ell-m-r-1}D^{r+1}\}$

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where A, B, C and D are defined as above.

Consider

$$(2.2.26) \quad Q = \min_{0 \le r \le k-m-2} \{A^{k+i-m}[\tfrac{B}{A}]^{r+1} - C^{k+i-m}[\tfrac{D}{C}]^{r+1}\}.$$

Now we want to show $\tfrac{B}{A} \ge \tfrac{D}{C}$.

Let $f(x) = G_n(x-d) - G_n^2(x-d)$ for $0 \le x \le d$.

$$\frac{df(x)}{dx} = g_n(x-d)[1-2G_n(x-d)] \ge 0, \quad 0 \le x \le d.$$

Since $0 \le x \le d$, $f(x) \ge f(0)$ and $G_n(t) = 1-G_n(-t)$, we can see that $\tfrac{B}{A} \ge \tfrac{D}{C}$.

By Lemma 2.2.1 and from (2.2.26),

$$(2.2.27) \quad Q = \min\{A^{k+i-m-1}B - C^{k+i-m-1}D, \; A^{i+1}B^{k-m-1} - C^{i+1}D^{k-m-1}\}.$$

From (2.2.25) and (2.2.27),

$$(2.2.28) \quad \inf_m \xi(2) = \min\{\min_{0 \le i \le m}[A^{k+i-m-1}B - C^{k+i-m-1}D],$$
$$\min_{0 \le i \le m}[A^{i+1}B^{k-m-1} - C^{i+1}D^{k-m-1}]\}.$$

By Lemma 2.2.1 and (2.2.28), we get

$$(2.2.29) \quad \inf_m \xi(2) = \min\{\min[A^{k-m-1}B - C^{k-m-1}D, A^{k-1}B - C^{k-1}D],$$
$$\min[AB^{k-m-1} - CD^{k-m-1}, A^{m+1}B^{k-m-1} - C^{m+1}D^{k-m-1}]\}.$$

This completes the proof.

Since $\xi = \sum_{m=0}^{k-1} \xi_m$, by Lemma 2.2.4 and Lemma 2.2.1, we can obtain the following result.

Corollary 2.2.2. Suppose that (i) $q_n(t) - g_n(-t) \ge 0$ for all $t \ge 0$ and (ii) the family of density functions $\{q_n(t-\cdot): \cdot \in \xi_1\}$ has a

monotone likelihood ratio, for every $n \geq 1$. For arbitrary fixed d_1, d_2 satisfying $d_1 = d_2 = d > \Delta$,

$$(2.2.30) \quad \inf_\Omega \alpha_{\underline{0}}(2) = \min\{A-C, A^k-B^k, AB^{k-1}-CD^{k-1}, A^{k-1}B-C^{k-1}D\}$$

where A, B, C and D are defined as in Lemma 2.2.4.

Now we wish to discuss the probability that I_1 excludes all the bad populations and the infimum of this probability over Ω.

Lemma 2.2.5. For $\underline{0} \in \Omega_m$,

$$(2.2.31) \quad \alpha_{\underline{0}}(3) \geq 1 - G_n^{k-m}(d_1) G(-\Delta + d_1)$$

Proof. For $\underline{0} \in \Omega_m$,

$$(2.2.32) \quad \alpha_{\underline{0}}(3) = P[0_{[i]} \notin I_1, \ i = 1,\ldots,m]$$

$$\geq P[0_{[1]} \geq T_{[k]} + d_2] + P[0_{[m]} \leq T_{[k]} - d_1]$$

$$= \prod_{j=1}^{k} G_n(0_{[1]} - 0_{[j]} - d_2) + 1 - \prod_{j=1}^{k} G_n(0_{[m]} - 0_{[j]} + d_1).$$

Let $\delta_i = 0_{[k]} - 0_{[i]}$, $i = 1,2,\ldots,k$.

$$(2.2.33) \quad \alpha_{\underline{0}}(3) = 1 + \prod_{j=1}^{k} G_n(\delta_j - \delta_1 - d_2) - \prod_{j=1}^{k} G_n(\delta_j - \delta_m + d_1)$$

since $\dfrac{\partial \alpha_{\underline{0}}(3)}{\partial \delta_1} \leq 0$ for $\delta_1 > \delta_2$, by letting $\delta_1 = \infty$, then

$$(2.2.34) \quad \alpha_{\underline{0}}(3) \geq 1 - \prod_{j=2}^{k} G_n(\delta_j - \delta_m + d_1).$$

Since the right hand side of (2.3.34) is nonincreasing in δ_j for $j = 2,3,\ldots,m-1, m+1,\ldots,k-1$, and is nondecreasing in δ_m, it follows

that the infimum of $1 - \prod\limits_{j=2}^{k} G_n(\delta_j - \delta_m + d_1)$ over the set $\{(\delta_2, \ldots, \delta_{k-1}):$

$0 < \delta_{k-1} < \ldots < \delta_{m+1} < \Delta < \delta_m < \ldots < \delta_2\}$ is achieved at $\delta_{m+1} = \ldots =$

$\delta_{k-1} = \Delta$, $\delta_m = \Delta$ and $\delta_2 = \ldots = \delta_{m-1} = \infty$. Thus

$$(2.2.35) \qquad \underline{\alpha}_0(3) \geq 1 - G_n^{k-m}(d_1) G_n(d_1 - \Delta)$$

The proof is complete.

Since $\Omega = \bigcup\limits_{m=0}^{k-1} \Omega_m$, by Lemma 2.2.5, we can easily lead to the following theorem.

Theorem 2.2.3. For arbitrary fixed d_1, d_2 satisfying $d_1 + d_2 > 0$,

$$(2.2.36) \qquad \underline{\alpha}_0(3) \geq 1 - G_n(d_1) G_n(-\Delta + d_1).$$

Remark 2.2.1. (i) The right hand side of (2.2.36) is independent of d_2. That is, for a given d_1 and for any fixed d_2 satisfying $d_1 + d_2 > 0$, $\underline{\alpha}_0(3)$ is bounded below by $1 - G_n(d_1) G_n(-\Delta + d_1)$. If the confidence interval is defined as $I_1 = (T_{[k]} - d_1, \infty)$ where d_1 is any given number, then $\inf\limits_{\Omega} \underline{\alpha}_0(3)$ can be obtained as follows.

$$(2.2.37) \qquad \inf_{\Omega} \underline{\alpha}_0(3) = 1 - G_n(d_1) G_n(-\Delta + d_1).$$

(ii) If we assume $\underline{\alpha}_\theta(4)$ is the probability that I_1 contains at least one good population and also I_1 excludes all bad populations, then $\underline{\alpha}_0(4) \geq \underline{\alpha}_0(1) + \underline{\alpha}_\theta(3) - 1$. From (2.2.11) and (2.2.36), we get

$$\underline{\alpha}_0(4) \geq \min\{G_n(d_1) - G_n(-d_2), G_n^k(d_1) - G_n^k(-d_2), G_n^{k-m}(d_1) G_n^m(\Delta + d_1) -$$
$$G_n^{k-m}(-d_2) G^m(\Delta - d_2), m = 1, \ldots, k-1\} - G(d_1) G_n(-\Delta + d_1).$$

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Similarly, we can find the lower bound of $\alpha_{\underline{\theta}}(5)$ which is defined as the probability that I_1 contains all good populations and also I_1 excludes all bad populations.

Let S' be the number of non-best (bad) populations that enter I_1. We are interested in $E(S')$, the expected size (over Ω) of non-best populations that enter I_1.

<u>Theorem 2.2.4.</u>

$$(2.2.38) \quad \sup_{\Omega} E(S') \leq (k-1)G_n(d_1)$$

Proof. For $\underline{\theta} \in \Omega_m$,

$$(2.2.39) \quad E(S) = \sum_{i=1}^{m} P[\theta_{[i]} \in I]$$

$$= \sum_{i=1}^{m} \{ \prod_{j=1}^{k} G_n(\theta_{[i]}-\theta_{[j]}+d_1) - \prod_{j=1}^{k} G_n(\theta_{[i]}-\theta_{[j]}-d_2) \}$$

$$\leq m\{ \prod_{j=1}^{k} G_n(\theta_{[k]}-\theta_{[j]}+d_1) - \prod_{j=1}^{k} G_n(\theta_{[1]}-\theta_{[j]}-d_2) \}$$

$$\leq m\{ \prod_{j=1}^{m} G_n(\theta_{[k]}-\theta_{[j]}+d_1) \prod_{j=m+1}^{k-1} G_n(\theta_{[k]}-\theta_{[j]}+d_1)G_n(d_1) -$$

$$G_n(-d_2)G_n^{k-1}[\theta_{[1]}-\theta_{[k]}-d_2] \}$$

$$\leq m\{G_n(d_1)G_n^{k-m-1}(\Lambda+d_1)G_n^m(\theta_{[k]}-\theta_{[1]}+d_1)-G_n(-d_2) \cdot$$

$$G_n^{k-1}[\theta_{[1]}-\theta_{[k]}-d_2] \}$$

$$\leq m\, G_n(d_1)G_n^{k-m-1}(\Lambda+d_1).$$

since $\Omega = \bigcup_{m=0}^{k-1} \Omega_m$, then

$$(2.2.40) \quad \sup_{\Omega} E(S') \leq \max\{m\, G_n(d_1)G_n^{k-m-1}(\Lambda+d_1)\ m=0,1,\ldots,k-1\}$$

$$= (k-1)G_n(d_1).$$

In the following, we deal with special values of d_1, d_2 and Λ.

<u>Case (i)</u>: Arbitrary but fixed d_1, d_2 satisfying $d_1 + d_2 > 0$ and $\Lambda = 0$. Since $\Lambda = 0$, the only good population is the one associated with $\theta_{[k]}$. Applying the earlier results,

$$(2.2.41) \quad \inf \alpha_2(1) = \inf \alpha_2(2) = \min\{G_n(d_1) - G_n(-d_2), G_n^k(d_1) - G_n^k(-d_2)\},$$

a result obtained by Dudewicz and Tong [26]. Also, in this case

$$(2.2.42) \qquad \alpha_2(3) = 1 - G_n^2(d_1).$$

<u>Case (ii)</u>: One side (lower) confidence interval. Arbitrary but fixed d_1 and $d_2 = \infty$ with $\Lambda > 0$. Our confidence interval is now $I_1 = (T_{[k]} - d_1, \infty)$. We can show that

$$(2.2.43) \quad \inf \alpha_2(1) = G_n^k(d_1),$$

$$(2.2.44) \quad \inf \alpha_2(2) = G_n(d_1) G_n^{k-1}(-\Lambda + d_1) \text{ and}$$

$$(2.2.45) \quad \inf \alpha_2(3) = 1 - G_n(d_1) G_n(-\Lambda + d_1).$$

<u>Case (iii)</u>: One side (upper) confidence interval. Arbitrary fixed d_2, $d_1 = \infty$ and $\Lambda > 0$. Now our confidence interval is defined as $I_1 = (-\infty, T_{[k]} + d_2)$, we can show that

$$(2.2.45) \quad \inf \alpha_2(1) = 1 - G(-d_2),$$

$$(2.2.46) \quad \inf \alpha_2(2) = 1 - G_n^{k-1}(\Lambda - d_2) G(-d_2) \text{ and}$$

$$(2.2.47) \quad \inf \alpha_2(3) = 0.$$

2.3. Result for the scale parameter case.

In this section, we assume θ_i is a scale parameter for T_i, $i = 1,\ldots,k$. It is assumed that there is no a priori knowledge about the θ_i's.

$$(2.3.1) \qquad G_n(t, \theta_i) = \begin{cases} G_n(\frac{t}{\theta_i}) & \text{for } t \geq 0 \\ 0 & \text{for } t < 0, \end{cases}$$

for $\theta_i > 0$, $k = 1,\ldots,k$.

For arbitrary fixed a, b satisfying $a \geq 1$ and $b \geq 1$, if the confidence interval is defined as

$$(2.3.2) \qquad I_2 = (\frac{1}{a} T_{[k]}, \, b \, T_{[k]})$$

where $T_{[k]} = \max_{1 \leq i \leq k} T_i$.

Let c be a given positive constant, $c \geq 1$. We say population θ_i is

$$(2.3.3) \qquad \text{good} \quad \text{if} \quad \theta_{[k]} \leq c \, \theta_i$$
$$\text{bad} \quad \text{if} \quad \theta_{[k]} > c \, \theta_i$$

where $\theta_{[k]} = \max_{1 \leq i \leq k} \theta_i$.

For given $\underline{\theta} = (\theta_1,\ldots,\theta_k)$ and $c \geq 1$, let $P_{\underline{\theta}}(1)$ denote the probability that I_2 contains at least one good population, let $P_{\underline{\theta}}(2)$ denote the probability that I_2 contains all good population and let $P_{\underline{\theta}}(3)$ denote the probability that I_2 excludes all the bad populations. Using the same arguments as in Section 2.2, we obtain the following results.

__Theorem 2.3.1.__ If the family of density functions $\{g_n(\frac{x}{\theta}): \theta \in \Lambda_2\}$ has monotone likelihood ratio, for every $n \geq 1$, then for arbitrary fixed a, b satisfying $a \geq 1$, $b \geq 1$ and $ab > \Lambda$, then

$$(2.3.4) \quad \inf_{\theta}(1) = \min\{G_n(a)-G_n(\tfrac{1}{b}),\ G_n^k(a) - G_n^k(\tfrac{1}{b}),$$

$$G_n^{k-m}(a)G^m(a\Lambda)-G_n^{k-m}(\tfrac{1}{b})G_n^m(\tfrac{\Lambda}{b}),\ m=1,\ldots,k-1\}.$$

__Corollary 2.3.1.__ Let the family of density functions $\{g_n(\frac{x}{\theta}):$ $\theta \in \Lambda_2\}$ has monotone likelihood ratio, for every $n \geq 1$. For arbitrary fixed a,b satisfying $a \geq 1$, $b \geq 1$, $g_n(\tfrac{1}{b}) \geq g_n(a)$ and $ab > \Lambda$,

$$(2.3.5) \quad \inf_{\theta}(1) = \min\{G_n(a)-G_n(\tfrac{1}{b}),G_n^k(a)-G_n^k(\tfrac{1}{b}),\ G_n(a)G_n^{k-1}(a\Lambda)-$$

$$G_n(\tfrac{1}{b})G_n^{k-1}(\tfrac{\Lambda}{b})\}.$$

__Theorem 2.3.2.__ If the family of density functions $\{g_n(\frac{x}{\theta}): \theta \in \Lambda_2\}$ has a monotone likelihood ratio, for every $n \geq 1$, then for arbitrary fixed a,b satisfying $a \geq 1$, $b \geq 1$ and $ab > \Lambda$,

$$(2.3.6) \quad \inf_{\theta}(2) = \min_{0 \leq m \leq k-1}\ \min_{0 \leq r \leq k-m-2}\ \min_{\theta}\{[G_n(a)]^{k+\theta-m-r-1}\cdot$$

$$G_n^{r+1}(a)-[G_n(\tfrac{\Lambda}{b})]^{k+\theta-m-r-1}G_n^{r+1}(\tfrac{1}{b})\}.$$

__Theorem 2.3.3.__ For arbitrary fixed $a \geq 1$ and $b \geq 1$,

$$(2.3.7) \quad \inf_{\theta}(3) = 1-G_n(a)G_n(\tfrac{a}{\theta}).$$

Let S' be the number of non-best (bad) populations that enter I_2. Also we can obtain the following result.

Theorem 2.3.4.

$$(2.3.8) \qquad \sup E(S') \leq (k-1)G_n(a).$$

In the following, we deal with some special values of d_1, d_2 and Δ.

Case (i): Arbitrary fixed a,b satisfying $a \geq 1$ and $b \geq 1$ and $\Delta = 1$. Since $\Delta = 1$ the only good population is the one associated with $\theta_{[k]}$. Applying the earlier results,

$$(2.3.9) \quad \inf P_\theta(1) = \inf P_\theta(2) = \min\{G_n(a)-G_n(\tfrac{1}{b}), G_n^k(a)-G_n^k(\tfrac{1}{b})\},$$

a result obtained by Saxena and Tong [66]. Also, in this case

$$(2.3.10) \qquad P_\theta(3) \geq 1-G_n^2(a).$$

Case (ii): One side (lower) confidence interval. Arbitrary but fixed $a \geq 1$ and $b = \infty$ with $\Delta \geq 1$. Our confidence interval is defined as $I_2 = (a\,T_{[k]}, \infty)$. We can show that

$$(2.3.11) \qquad \inf P_\theta(1) = G_n^k(a),$$

$$(2.3.12) \qquad \inf P_\theta(2) = G_n(a)G_n^{k-1}(\tfrac{a}{\Delta}) \text{ and}$$

$$(2.3.13) \qquad \inf P_\theta(3) = 1-G_n(a)G_n(\tfrac{a}{\Delta}).$$

Case (iii): One side (upper) confidence interval. For arbitrary fixed b, $a = \infty$ and $\Delta \geq 1$. Now our confidence interval is defined as

$$I_2 = (0, b\,T_{[k]}).$$

We can show that

$$(2.3.14) \qquad \inf_{\underline{\ }} (1) = 1-G_n(\tfrac{1}{b}),$$

$$(2.3.15) \qquad \inf_{\underline{\ }} (2) = 1-G_n(\tfrac{1}{b})G_n^{k-1}(\tfrac{\cdot}{b}) \text{ and}$$

$$(2.3.16) \qquad \inf_{\underline{\ }} (3) = 0.$$

2.4. Some examples.

We shall discuss the above problem for the means of normal populations and for the scale parameter of the gamma population.

Let $\pi_1, \pi_2, \ldots, \pi_k$ be k normal populations with unknown means $\mu_1, \mu_2, \ldots, \mu_k$ and a common known variance σ^2. Let $\bar{X}_i$ be the sample mean from π_i based on a sample of size n. Let

$$(2.4.1) \quad T_i = \bar{X}_i \quad \text{and} \quad T_{[k]} = \max_{1 \le i \le k} \bar{X}_i.$$

Let $G_n(x, \mu_i)$ be the c.d.f. of $\bar{X}_i$ from π_i, i = 1,...,k. In this case, $G_n(x, \mu_i) = G_n(x-\mu_i)$ and

$$(2.4.2) \qquad G_n(x) = \Phi(\tfrac{\sqrt{n}}{\sigma} x)$$

where Φ is the standard normal distribution function. The confidence interval I_1 is of the form $(T_{[k]}-d_1, T_{[k]}+d_2)$, where $d_1+d_1 > 0$. In this case, applying Theorem 2.2.1, we get

$$(2.4.3) \quad \inf_{\underline{\ }} (1) = \min\{A-B,\ A^k-B^k,\ A^{k-m}C^m-B^{k-m}D^m,\ m=1,2,\ldots,k-1\}$$

where

$$A = \Phi(\tfrac{\sqrt{n}\, d_1}{\sigma}), \quad B = \Phi(\tfrac{-\sqrt{n}\, d_2}{\sigma}), \quad C = \Phi(\tfrac{\sqrt{n}(\delta+d_1)}{\sigma}) \text{ and } D = \Phi(\tfrac{\sqrt{n}(\delta-d_2)}{\sigma}).$$

For a given P^*, $\Lambda > 0$, $d_1 > 0$, $d_2 > 0$ and $d_1 + d_2 > \Delta$, since $\lim_{n \to \infty} A = 1$, $\lim_{n \to \infty} B = 0$, $\lim_{n \to \infty} C = 1$ and $\lim_{n \to \infty} D = 0$, there exists a smallest integer N such that for every $n \geq N$, we have $\inf_{\Omega} \alpha_\theta(1) \geq P^*$. Let x_1 be the unique solution of $\phi(d_1 x) - \phi(-d_2 x) = P^*$. Let x_2 be the unique solution of $\phi^k(d_1 x) - \phi^k(-d_2 x) = P^*$. Let y_m be the unique solution of

$$\phi^{k-m}(d_1 x)\phi^m((\Lambda+d_1)x) - \phi^{k-m}(-d_2 x)\phi^m((\Lambda-d_2)x) = P^*, \quad m=1,2,\ldots,k-1.$$

Let

$$N_i = [\sigma^2 x_i^2], \quad i = 1,2 \quad \text{and} \quad M_i = [\sigma^2 y_i^2], \quad i = 1,\ldots,k-1,$$

where $[x]$ denotes the smallest integer $\geq x$. It follows that the smallest sample size N required to satisfy $\inf_{\Omega} \alpha_0(1) \geq P^*$ is given by

$$(2.4.4) \qquad N = \min\{N_i, M_j, \ i=1,2, \ j=1,\ldots,k-1\}.$$

Consider the probability that I_1 contains all good populations. For given P^*, $\Lambda > 0$, $d_1 > \Lambda$ and $d_2 > 0$, then

$$\lim_{n \to \infty} G_n(d_1) = 1, \quad \lim_{n \to \infty} G_n(-\Lambda+d_1) = 1 \quad \text{and} \quad \lim_{n \to \infty} G_n(-d_2) = 0.$$

By Theorem 2.2.2, we can always find the smallest sample size N which satisfies

$$(2.4.5) \qquad\qquad \inf_{\Omega} \alpha_0(2) \geq P^*.$$

Consider the probability that I_1 excludes all good populations. For given P^*, $\Lambda > 0$, d_1, d_2 such that $d_1+d_2 > 0$ and $d_1 < \Lambda$, then

$$\lim_{n \to \infty} G(-\Lambda+d_1) = 0.$$

By Theorem 2.2.3, we can always find the smallest sample size N required to satisfy

(2.4.6) $\qquad \inf_{t_c}(3) \quad D^\star.$

Now let us consider the problem in relation to the scale parameters of the populations. Let $\pi_1, \pi_2, \ldots, \pi_k$ be k gamma distributions. The distribution associated with π_i has the density function $g(x, \alpha, \theta_i)$ where

(2.4.7)
$$g(x, \alpha, \theta_i) = \begin{cases} \dfrac{1}{\Gamma(\alpha)\theta_i^\alpha} x^{\alpha-1} e^{-\frac{x}{\theta_i}} & \text{for } x > 0 \\ \\ 0 & \text{otherwise.} \end{cases}$$

We assume that the shape parameter α is the same for all populations and α is known. Let T_i be the sample sum from π_i based on n observations and let $T_{[k]} = \max_{1 \le i \le k} T_i$. Then the confidence interval is given by

(2.4.8) $\qquad I_2 = (\frac{1}{a} T_{[k]}, \, b\, T_{[k]})$ where $ab > 1$.

Let $G_n(x, \theta_i)$ be the c.d.f. of T_i from $i = 1, \ldots, k$.
In this case, $G_n(x, \theta_i) = G_n(\frac{x}{\theta_i})$ and

(2.4.9) $\qquad G_n(x)$ is the c.d.f. of a gamma distribution with scale parameter 1 and shape parameter $n\alpha$. Since the condition in Theorem 2.3.1 holds, we get

(2.4.10) $\inf_{\theta}(1) = \min\{G(a)-G(\frac{1}{b}), \; G^k(a)-G^k(\frac{1}{b}),$

$\qquad\qquad\qquad G^{k-m}(a)G^m(a)-G^{k-m}(\frac{1}{b})G^m(\frac{1}{b}), m=1,\ldots,k-1\}.$

For given P^*, ≥ 1, $a > 1$, $b > 1$ with $ab > 1$, since $\lim\limits_{n \to \infty} G_n(a) = 1$, $\lim\limits_{n \to \infty} G_n(\frac{1}{b}) = 0$ and $\lim\limits_{n \to \infty} G_n(a) = 1$, there exists a smallest integer N such that for every $n \geq N$, we have $\inf \gamma_n(1) \geq P^*$.

Let N_1 be the smallest positive integer such that $G_n(a) - G_n(\frac{1}{b}) \geq P^*$. Let N_2 be the smallest positive integer such that $G_n^k(a) - G_n^k(\frac{1}{b}) \geq P^*$. Let M_m be the smallest positive integer such that $G_n^{k-m}(a) G^m(a) - G^{k-m}(\frac{1}{b}) G^m(\frac{1}{b})$, $m = 1, 2, \ldots, k-1.$.

It follows that the smallest sample size N required to satisfy $\inf \gamma_n(1) \geq P^*$ is given by

$$(2.4.11) \qquad N = \min\{N_i, M_j, i=1,2, j=1,2,\ldots,k-1\}.$$

Consider the probability that I_2 contains all good populations. For a given P^*, ≥ 1, $a >$ and $b > 1$, then

$$\lim_{n \to \infty} G_n(a) = 1, \quad \lim_{n \to \infty} G_n(\frac{a}{}) = 1 \text{ and } \lim_{n \to \infty} (\frac{1}{b}) = 0.$$

By Theorem 2.3.2, we can always find the smallest sample size N required to satisfy $\inf \gamma_n(2) \geq P^*$.

Consider the probability that I_2 excludes all good populations. For given P^*, 1, b and a with $a < $, then $\lim\limits_{n \to \infty} G_n(\frac{a}{}) = 0$. By Theorem 2.3.3, we can always find the smallest sample size N required to satisfy $\inf \gamma_n(3) \geq P^*$.

Remark 2.3.1. (i) If the sample size N is preassigned and P^* and are given, then the confidence interval I_1 can always be found from (2.2.11), (2.2.23) and (2.2.36) with $G_n(x)$ as defined in (2.4.2) so that $\inf \gamma_n(i) \geq P^*$, $i = 1,2,3$.

Similarly, the confidence interval I_2 can always be obtained from (2.3.4), (2.3.6) and (2.3.7) with $G_n(x)$ as defined in (2.4.9) so that $\inf \epsilon_n(i) \geq P^*$, $i = 1,2,3$.

(ii) In location parameter case, if the population with the smallest θ-value is called the best population. Let τ be given, we say population π_i is

(2.4.12) good if $\theta_i < \theta_{[1]} + \tau$,

bad if $\theta_i \geq \theta_{[1]} + \tau$

where $\theta_{[1]} = \min_{1 \leq i \leq k} \theta_i$.

Then the confidence interval is given by

(2.4.13) $I_1 = (T_{[1]} - d_1, T_{[1]} + d_2)$

where $T_{[1]} = \min_{1 \leq i \leq k} T_i$.

The calculation of the coverage probabilities $\epsilon_n(i)$, $i = 1,2,3$, and their infima can be handled similarly as in Section 2.2. The scale parameter case can be handled in a similar manner.

CHAPTER III

SELECTION PROCEDURES IN TERMS OF

MAJORIZATION AND WEAK MAJORIZATION

3.1. Introduction

Over the last fifty years, majorization and Schur functions have been applied to develop many useful inequalities in many branches of mathematics. Some of the references are: Schur [67], Ostrowski [57], Hardy, Littlewood and Pólya [45], Beckenbach and Bellman [17], Marshall and Proschan [52] and Marshall and Olkin [50]. In recent times the techniques of majorization and Schur functions have been applied to probability and statistics (see, for example, Marshall, Olkin and Proschan [51], Rinott [62], Proschan and Sethuraman [61], Nevius, Proschan and Sethuraman [55] and Gupta and Wong [44]).

Recently, Nevius, Proschan and Sethuraman [56] introduced a stochastic version of weak majorization. They have discussed some properties and made some applications.

Gupta [32] defined a class of selection procedures and considered some of its properties. Some additional results concerning the properties of this class of procedures were obtained by Gupta and Panchapakesan [36], Gupta and

Panchapkesan [37] defined a class of selection procedures which
is a natural generalization of the class considered by Gupta [32].
In that paper, they obtained a sufficient condition for the
probability of a correct selection to be nondecreasing in λ when
$\lambda_1 = \ldots = \lambda_k = \lambda$, where λ_i is the parameter associated with π_i,
$i = 1,\ldots,k$. They also obtained the supremum of the expected
size of the selected subset and showed that if the sufficient
condition holds, it takes place when the λ_i's are equal.

In this chapter, we order the parameter space by means of
majorization or weak majorization and propose some selection
procedures when the parameter $\underline{\lambda}_i$ associated with π_i, $i=1,\ldots,k$
is a vector. Section 3.2 defines a class of procedures R_h for
selecting the population associated with vector $\underline{\lambda}_{[k]}$ (see
Section 3.2 for definitions). A sufficient condition is obtained
for the infimum of the probability of a correct selection to be
Schur-convex in $\underline{\lambda}$. Also another sufficient condition is obtained
for the supremum of the expected size of the selected subset to
take place when $\underline{\lambda}_1 = \ldots = \underline{\lambda}_k$. Some special cases of interest are
discussed. In Section 3.3, a sufficient condition is obtained
for the same infimum of the probability of a correct selection
to be nondecreasing and Schur-convex in $\underline{\lambda}$. Section 3.4 defines
a class of procedure R_H for the selection of the population with
vector $\underline{\lambda}_{[1]}$. Some properties of the selection procedure are
briefly discussed. Section 3.5 and 3.6 deal with selection procedures
for multivariate normal distributions in terms of majorization

and weak majorization of mean vectors. Various cases corresponding
to the known or unknown common covariance matrix Σ are studied.
Properties of these selection procedures are also established.

3.2 A class of selection procedures R_h (and some properties)

for vector-valued $\underline{\lambda}_i$ in terms of majorization

First we give the definitions of majorization, Schur-concave
and Schur-convex functions.

Definition 3.2.1. A vector $\underline{a} = (a_1,\ldots,a_p)$ is said to majorize
a vector $\underline{b} = (b_1,\ldots,b_p)$, if $a_1 \geq \ldots \geq a_p$, $b_1 \geq \ldots \geq b_p$, and
$\sum_{i=1}^{r} a_i \geq \sum_{i=1}^{r} b_i$, $r = 1,\ldots,p-1$, while $\sum_{i=1}^{p} a_i = \sum_{i=1}^{p} b_i$; we write,
$\underline{a} \underset{m}{>} \underline{b}$.

The above definition is according to Beckenbach and Bellman
[17] and differs slightly from that of Hardy, Littlewood and
Pólya [45].

Definition 3.2.2. H is called a Schur-concave (Schur-convex)
function if $H(\underline{a}) \leq H(\underline{b})$ $(H(\underline{a}) \geq H(\underline{b}))$ whenever $\underline{a} \underset{m}{>} \underline{b}$. A Schur
function is a function which in either Schur-concave or Schur-
convex.

From above definitions, we know that majorization is a partial
ordering in R^p, the p-dimensional Euclidean space and Schur functions
are functions that are monotone with respect to this partial
ordering. Now we state the following theorem.

Theorem 3.2.1. (Ostrowski [57])

Assume H is defined for $Z_1, \ldots, Z_p$ and first partial derivatives of H exist. Then $H(\underline{Z}) \geq H(\underline{Z}')$ for all $\underline{Z} \underset{m}{\succ} \underline{Z}'$ if and only if

$$(3.2.1) \qquad \frac{\partial H}{\partial Z_i} - \frac{\partial H}{\partial Z_j} \geq 0, \text{ for } i < j, \; i, j = 1, \ldots, p.$$

Let $\pi_1, \ldots, \pi_k$ be k populations. Let A be an interval on the real line. Associated with π_i is a vector $\underline{X}_i = (X_{i1}, \ldots, X_{ip})$, $i = 1, \ldots, k$ where X_{ij} has density $f(\lambda_{ij}, x)$, $i = 1, \ldots, k$, $j = 1, \ldots, p$. We assume that $\lambda_{ij} \subseteq A$ and $\lambda_{i1} \leq \ldots \leq \lambda_{ip}$, $i = 1, \ldots, k$. We say that π_i is better than π_j if $\underline{\lambda}_i = (\lambda_{i1}, \ldots, \lambda_{ip}) \underset{m}{\succ} \underline{\lambda}_j = (\lambda_{j1}, \ldots, \lambda_{jp})$. It is assumed that among the k given populations, there always exists one population which is better than others. This is equivalent to saying that there exists a $\underline{\lambda}_{[k]}$ such that $\underline{\lambda}_{[k]} \underset{m}{\succ} \underline{\lambda}_i$ for all $i = 1, \ldots, k$. This population is called the best population. If there are more than one "best" population, then we assume that one of them is tagged as the best. Based on one observation $X_{i1}, \ldots, X_{ip}$ from population π_i, we construct a suitable statistic $T_i = g(X_{i1}, \ldots, X_{ip})$, $i = 1, \ldots, k$. Let $F_{\underline{\lambda}_i}(x) = P[T_i \leq x]$ be the distribution function of T_i from population π_i. Based on the values of T_i from π_i, $i = 1, \ldots, k$, we wish to define a class of procedures for selecting a non-empty subset of the k populations such that the probability is at least P^* ($\frac{1}{k} < P^* < 1$) that the population associated with $\underline{\lambda}_{[k]}$ is included in the selected subset.

Let

(3.2.2) $E = \{\underline{\lambda} = (\lambda_1,\ldots,\lambda_p): \lambda_1 \geq \cdots \geq \lambda_p, \lambda_i \in \Lambda, i=1,\ldots,p\}$

and

(3.2.3) $\Omega = \{\underline{\omega} = (\underline{\lambda}_1,\ldots,\underline{\lambda}_k): \underline{\lambda}_i \in E, i=1,\ldots,k$ and there exists

some i such that $\underline{\lambda}_i \underset{m}{>} \underline{\lambda}_j$ for each j}.

We wish to define a selection rule R such that

(3.2.4) $\inf_{\Omega} P[CS|R] \geq P^{*}.$

Let $h = h_{c,d}$; $c \in [1,\infty)$, $d \in [0,\infty)$ be a function defined on the real line satisfying the following properties: for every real x,

(3.2.5)   (i) $h_{c,d}(x) \geq x$

           (ii) $h_{1,0}(x) = x$

          (iii) $h_{c,d}(x)$ is continuous in c and d

           (iv) $h_{c,d}(x) \uparrow \infty$ as $d \to \infty$ and/or $xh_{c,d}(x) \uparrow \infty$ as $c \to \infty$, $x \neq 0$.

Some functions satisfying these properties that will be of interest are cx, x+d and cx+d. Now, we define a class of procedures R_h as follows.

     R_h: Select population π_i if and only if

(3.2.6) $h(T_i) \geq \max_{1 \leq r \leq k} T_r.$

Because of (3.2.5)-(i), the procedure R_h will always select a

non-empty subset. Let $T_{(i)}$ be the random variable associated with $\underline{\lambda}_{[i]}$ and let $F_{\underline{\lambda}_{[i]}}$ be the cumulative distribution function (c.d.f.) of $T_{(i)}$, $i=1,\ldots,k$, we have

$$(3.2.7) \quad P[CS|R_h] = P[h(T_{(k)}) \geq T_{(i)}, \; i=1,\ldots,k-1]$$

$$= \int_{-\infty}^{\infty} \prod_{i=1}^{k-1} F_{\underline{\lambda}_{[i]}}(h(x)) dF_{\underline{\lambda}_{[k]}}(x).$$

We now assume that the distributions are Schur-concave in $\underline{\lambda}$, i.e., for $\underline{\lambda} \underset{m}{>} \underline{\lambda}'$, $F_{\underline{\lambda}}$ and $F_{\underline{\lambda}'}$ are distinct and

$$(3.2.8) \qquad F_{\underline{\lambda}}(x) \leq F_{\underline{\lambda}'}(x), \quad \text{for all } x.$$

Then

$$P[CS|R_h] \geq \int_{-\infty}^{\infty} F_{\underline{\lambda}_{[k]}}^{k-1}(h(x)) dF_{\underline{\lambda}_{[k]}}(x).$$

Hence

$$(3.2.9) \qquad \inf_{\Omega} P[CS|R_h] = \inf_{\underline{\lambda}} \psi(\underline{\lambda}; c,d,k)$$

where

$$(3.2.10) \qquad \psi(\underline{\lambda};c,d,k) = \int_{-\infty}^{\infty} F_{\underline{\lambda}}^{k-1}(h(x)) dF_{\underline{\lambda}}(x), \quad \underline{\lambda} \in E.$$

In the following theorem, we are interested in a sufficient condition for the Schur-convexity of $E_\psi(T,\underline{\lambda})$ where $\psi(x,\underline{\lambda})$ is some real-valued function, $\underline{\lambda}=(\lambda_1,\ldots,\lambda_p) \in E$ and $T=g(X_{\lambda_1},\ldots,X_{\lambda_p})$ is a function of $X_{\lambda_1},\ldots,X_{\lambda_p}$. Let $F_{\underline{\lambda}}(x) = P[T \leq x]$.

Theorem 3.2.2. Let $\{F_{\underline{\lambda}}(x), \; \underline{\lambda} \in E\}$ be a family of continuous distributions on the real line such that $F_{\underline{\lambda}}(x)$ is a differentiable function in x and λ_i and let $\psi(x,\underline{\lambda})$ be a bounded real-valued and differentiable function in x and λ_i, $i = 1,\ldots,p$. Then $E_\psi(T,\underline{\lambda})$ is Schur-convex in $\underline{\lambda} \in E$ provided that

$$(3.2.11) \quad \begin{vmatrix} \frac{\partial}{\partial x} F_{\underline{\lambda}}(x) & \frac{\partial}{\partial x} \psi(x,\underline{\lambda}) \\ (\frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j}) F_{\underline{\lambda}}(x) & (\frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j}) \psi(x,\underline{\lambda}) \end{vmatrix} \geq 0, \text{ for } i \neq j, i,$$
$$j = 1,..,P.$$

Proof. We argue along the lines of the proof of Theorem 2.1 of Gupta and Panchapkesan [37]. We assume that $F_{\underline{\lambda}}(x)$, $\underline{\lambda} \in E$ has the support I. Let

$$(3.2.12) \quad A(\underline{\lambda}) = E_{\underline{\lambda}}(T,\underline{\lambda}) = \int_I \psi(x,\underline{\lambda}) dF_{\underline{\lambda}}(x).$$

Consider $\underline{u}_1$, $\underline{u}_2 \in E$, where $\underline{u}_i = (u_{i1},...,u_{ip})$, $i = 1,2$, $u_{i1} \geq u_{i2} \geq ... \geq u_{ip}$, $i = 1,2$ and assume $\underline{u}_1 = \underline{u}_2$. Define

$$(3.2.13) \quad A_1(\underline{u}_1,\underline{u}_2) = \int_I \psi(x,\underline{u}_2) dF_{\underline{u}_1}(x),$$

$$(3.2.14) \quad A_2(\underline{u}_1,\underline{u}_2) = \int_I \psi(x,\underline{u}_1) dF_{\underline{u}_2}(x),$$

and

$$(3.2.15) \quad B(\underline{u}_1,\underline{u}_2) = A_1(\underline{u}_1,\underline{u}_2) + A_2(\underline{u}_1,\underline{u}_2).$$

By integrating $A_1(\underline{u}_1,\underline{u}_2)$ by part (see Gupta and Panchapkesan [37]),

$$(3.2.16) \quad B(\underline{u}_1,\underline{u}_2) = \text{a term independent of } \underline{u}_1$$
$$+ \int_I [\psi(x,\underline{u}_1) f_{\underline{u}_2}(x) - \psi'(x,\underline{u}_2) F_{\underline{u}_1}(x)] dx$$

where $\psi'(x,\underline{u}) = \frac{\partial}{\partial x} \psi(x,\underline{u})$ and $f_{\underline{u}}(x) = \frac{d}{dx} F_{\underline{u}}(x)$.

For $i \neq j$, we have

$$(3.2.17) \quad (\frac{\partial}{\partial u_{1i}} - \frac{\partial}{\partial u_{1j}}) B(\underline{u}_1,\underline{u}_2) = \int_I f_{\underline{u}_2}(x) [\frac{\partial \psi(x,\underline{u}_1)}{\partial u_{1i}} - \frac{\partial \psi(x,\underline{u}_1)}{\partial u_{1j}}]$$
$$- \psi'(x,\underline{u}_2)[\frac{\partial F_{\underline{u}_1}(x)}{\partial u_{1i}} - \frac{\partial F_{\underline{u}_1}(x)}{\partial u_{1j}}] dx$$

We can show that, for $i = 1,\ldots,p$,

$$(3.2.18) \quad \frac{\partial}{\partial \lambda_i} B(\underline{\lambda},\underline{\lambda}) = \sum_{j=1}^{2} \frac{\partial}{\partial u_{j i}} B(\underline{u}_1,\underline{u}_2) \Big|_{\underline{u}_1 = \underline{u}_2 = \underline{\lambda}}$$

where $\underline{\lambda} = (\lambda_1,\ldots,\lambda_p)$ and $\underline{u}_1,\ldots,\underline{u}_2$

From (3.2.18) we get

$$(3.2.19) \quad \frac{\partial}{\partial \lambda_i} B(\underline{\lambda},\underline{\lambda}) = 2 \cdot \frac{\partial}{\partial u_{1i}} B(\underline{u}_1,\underline{u}_2) \Big|_{\underline{u}_1 = \underline{u}_2 = \underline{\lambda}}$$

and

$$(3.2.20) \quad \frac{\partial}{\partial \lambda_j} B(\underline{\lambda},\underline{\lambda}) = 2 \frac{\partial}{\partial u_{1j}} B(\underline{u}_1,\underline{u}_2) \Big|_{\underline{u}_1 = \underline{u}_2 = \underline{\lambda}}.$$

From (3.2.19) and (3.2.20), we get

$$(3.2.21) \quad \left(\frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j}\right) B(\underline{\lambda},\underline{\lambda}) = 2\left(\frac{\partial}{\partial u_{1i}} - \frac{\partial}{\partial u_{1j}}\right) B(\underline{u}_1,\underline{u}_2) \Big|_{\underline{u}_1 = \underline{u}_2 = \underline{\lambda}}.$$

If

$$(3.2.22) \quad \begin{vmatrix} \frac{\partial}{\partial x} F_{\underline{u}_2}(x) & \frac{\partial}{\partial x} \psi(x,\underline{u}_2) \\ \left(\frac{\partial}{\partial u_{1i}} - \frac{\partial}{\partial u_{1j}}\right) F_{\underline{u}_1}(x) & \left(\frac{\partial}{\partial u_{1i}} - \frac{\partial}{\partial u_{1j}}\right) \psi(x,\underline{u}_1) \end{vmatrix} \geq 0, \quad \begin{array}{l} \forall \, i < j, \\ i,j = 1,\ldots p, \end{array}$$

then from (3.2.17),

$$(3.2.23) \quad \left(\frac{\partial}{\partial u_{1i}} - \frac{\partial}{\partial u_{1j}}\right) B(\underline{u}_1,\underline{u}_2) \geq 0 \quad \text{for } i < j, \ i,j = 1,\ldots,p.$$

Hence if (3.2.22) holds, we have from (3.2.21)

$$(3.2.24) \quad \left(\frac{\partial}{\partial \lambda_i} - \frac{\partial}{\partial \lambda_j}\right) B(\underline{\lambda},\underline{\lambda}) \geq 0 \quad \text{for } i < j, \ i,j = 1,\ldots,p.$$

Note that

$$(3.2.25) \quad B(\underline{\lambda},\underline{\lambda}) = 2 A(\underline{\lambda}).$$

Applying Theorem 3.2.1 and from (3.2.24),

it follows that $B(\underline{\lambda},\underline{\lambda})$ in Schur-convex in $\underline{\lambda}$ if (3.2.22)
holds. Since $\underline{u}_1 = \underline{u}_2$, then $A(\underline{\lambda})$ is Schur-convex in $\underline{\lambda}$ if (3.2.11)
holds.

This completes the proof of the theorem.

In some cases, we will be dealing with the function $\psi(x,\underline{\lambda})$
such that it satisfies

$$(3.2.26) \qquad \frac{\partial}{\partial\lambda_1}\psi(x,\underline{\lambda}) = \ldots = \frac{\partial}{\partial\lambda_p}\psi(x,\underline{\lambda}),$$

then $(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})\psi(x,\underline{\lambda}) = 0$, $i < j$, $i,j = 1,\ldots,p$. Hence
we have the following result.

<u>Corollary 3.2.1.</u> If $\psi(x,\underline{\lambda})$ satisfies (3.2.26), then $E\psi(T,\underline{\lambda})$ is
Schur-convex in $\underline{\lambda}$ if

$$(3.2.27) \qquad \cdot(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})F_{\underline{\lambda}}(x) \cdot \frac{\partial}{\partial x}\psi(x,\underline{\lambda}) \leq 0, \ i < j, \ i, \ j=1,\ldots,p.$$

<u>Corollary 3.2.2.</u> If $F_{\underline{\lambda}}(x)$ is Schur-concave in $\underline{\lambda} \in E$, $\psi(x,\underline{\lambda})$
satisfies (3.2.26) and $\psi(x,\underline{\lambda})$ is nondecreasing in x, then $E\psi(T,\underline{\lambda})$
is Schur-convex in $\underline{\lambda} \in E$.

Proof. Since $F_{\underline{\lambda}}(x)$ is Schur-concave in $\underline{\lambda} \subset E$,

$$(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})F_{\underline{\lambda}}(x) \leq 0, \ \text{for} \ i < j, \ i, \ j=1,\ldots,p.$$

Also $\psi(x,\underline{\lambda})$ is nondecreasing in x, then $\frac{\partial}{\partial x}\psi(x,\underline{\lambda}) \geq 0$.
Hence by Corollary 3.2.1, the result follows.

As a special case, if $\psi(x,\underline{\lambda}) = \psi(x)$, then we have the following
result.

<u>Corollary 3.2.3.</u> If $F_\lambda(x)$ is Schur-concave in $\underline{\lambda}$ and $\psi(x)$ is nondecreasing in x, then $E\psi(T)$ is Schur-convex in $\underline{\lambda}$.

<u>Theorem 3.2.3.</u> For the procedure R_h, let $\{F_\lambda(x), \underline{\lambda} \in E\}$ be a family of continuous distributions on the real line such that $F_\lambda(x)$ and $F_\lambda(h(x))$ are differentiable functions in x and λ_i, $i = 1,\ldots,p$. Then $\psi(\underline{\lambda};c,d,k)$ as defined in (3.2.10) is Schur-convex in $\underline{\lambda} \in E$ provided that

$$(3.2.28) \quad f_{\underline{\lambda}}(x)(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})F_{\underline{\lambda}}(h(x)) - h'(x)f_{\underline{\lambda}}(h(x))(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})F_{\underline{\lambda}}(x) \geq 0,$$

$$\text{for } i < j, \ i,j = 1,\ldots,p,$$

where $f_\lambda(x) = \frac{d}{dx} F_\lambda(x)$ and $h'(x) = \frac{d}{dx} h(x)$.

Proof. By letting $\psi(x,\underline{\lambda}) = F_\lambda^{k-1}(h(x))$ and using Theorem 3.2.2. yields the proof.

We note that

$$(3.2.29) \qquad \frac{\partial}{\partial\lambda_1} \psi(\underline{\lambda};c,d,k) = \ldots = \frac{\partial}{\partial\lambda_p} \psi(\underline{\lambda};c,d,k), \text{ if}$$

$$(3.2.30) \quad f_{\underline{\lambda}}(x)(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})F_{\underline{\lambda}}(h(x)) - h'(x)f_\lambda(h(x))(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})F_{\underline{\lambda}}(x) = 0,$$

$$\text{for } i < j, \ i,j = 1,\ldots,k.$$

<u>Case (i):</u> $F_\lambda(x) = F(x-\xi(\underline{\lambda})), \ -\infty < \xi(\underline{\lambda}) < \infty$ and $h(x) = x+d, \ d > 0$ where $\xi(\underline{\lambda})$ is a Schur-convex in $\underline{\lambda}$, then

$$(3.2.31) \qquad \frac{\partial}{\partial\lambda_i} F_\lambda(x) = -f(x-\xi(\underline{\lambda})) \frac{\partial\xi(\underline{\lambda})}{\partial\lambda_i} = -f_{\underline{\lambda}}(x) \frac{\partial\xi(\lambda)}{\partial\lambda_i}.$$

In this case, (3.2.30) is satisfied and $\psi(\underline{\lambda};c,d,k)$ satisfies (3.2.29).

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<u>Case (ii)</u>: If $F_\lambda(x) = F(\frac{x}{\eta(\lambda)})$, $x > 0$, $\eta(\lambda) > 0$ and $h(x) = cx$,

$c \geq 1$ where $\eta(\lambda)$ is a Schur-convex in λ, then

$$(3.2.32) \quad \frac{\partial F_\lambda(x)}{\partial \lambda_i} = f(\frac{x}{\eta(\lambda)})[- \frac{x}{\eta^2(\lambda)} \frac{\partial}{\partial \lambda_i} \eta(\lambda)] = - \frac{x}{\eta(\lambda)} f_\lambda(x) \frac{\partial \eta(\lambda)}{\partial \lambda_i}.$$

In this case, (3.2.30) is satisfied and $\psi(\lambda;c,d,k)$ satisfies (3.2.29).

Actually, in the above two cases, $\psi(\lambda;c,d,k)$ does not involve λ, i.e., $\psi(\lambda;c,d,k)$ is independent of λ.

Let S denote the size of the subset selected by the procedure R_h. Let $\Omega' = \{\omega = (\lambda_1,\ldots,\lambda_k): \lambda_i \in E, i=1,\ldots,k$ and there exist $\ell_1,\ldots,\ell_k$ such that $\lambda_{\ell_1} \overset{>}{m} \lambda_{\ell_2} \overset{>}{m} \cdots \overset{>}{m} \lambda_{\ell_k}\}$.

Let $E_\omega(S|R_h)$ be the expected size of the selected subset using R_h, $\omega \in \Omega'$. Let $\lambda_{[1]} \overset{<}{m} \lambda_{[2]} \overset{<}{m} \cdots \overset{<}{m} \lambda_{[k]}$, for $\omega \in \Omega'$. It is easy to see that

$$(3.2.33) \quad E_\omega(S|R_h) = \sum_{i=1}^k p_i,$$

where

$$(3.2.34) \quad p_i = \int \prod_{\substack{r=1 \\ r \neq i}}^k F_{\lambda_{[r]}}(h(x))dF_{\lambda_{[i]}}(x),$$

and $F_{\lambda_{[i]}}$ is the c.d.f. of $T_{(i)}$ which is associated with $\lambda_{[i]}$, $i=1,\ldots,k$. Using the same arguments as in Gupta and Panchapkesan [37] and Theorem 3.2.2, we have the following theorem.

<u>Theorem 3.2.4.</u> Let $\{F_\lambda(x), \lambda \in E\}$ and $\{F_\lambda(h(x)), \lambda \in E\}$ be as in the hypothesis of Theorem 3.2.3. For $\omega \in \Omega'$, $E_\omega(S|R_h)$ is Schur-convex

in $\underline{\lambda}$, where $\underline{\lambda}_{[1]} = \ldots = \underline{\lambda}_{[m]} = \underline{\lambda} \underset{m}{<} \underline{\lambda}_{[m+1]} \underset{m}{<} \cdots \underset{m}{<} \underline{\lambda}_{[k]}$ and consequently

$\sup\limits_{\Omega'} E(S|R_h)$ takes place at $\underline{V}$ where $\underline{\lambda}_{[1]} = \ldots = \underline{\lambda}_{[k]} = \underline{V}$ provided that,

for $\underline{\lambda}_1 \underset{m}{<} \underline{\lambda}_2$ and $\underline{\lambda}_1, \underline{\lambda}_2 \in E$, the following holds;

$$(3.2.35) \quad (\frac{\partial}{\partial \lambda_{1i}} - \frac{\partial}{\partial \lambda_{1j}})F_{\underline{\lambda}_1}(h(x))f_{\underline{\lambda}_2}(x) - (\frac{\partial}{\partial \lambda_{1i}} - \frac{\partial}{\partial \lambda_{1j}})F_{\underline{\lambda}_1}(x)f_{\underline{\lambda}_2}(h(x)) \cdot$$
$$h'(x) \geq 0,$$

for $i < j$, $i,j = 1,\ldots,p$ where $\underline{\lambda}_i = (\lambda_{i1},\ldots,\lambda_{ip})$, $i=1,2$.

<u>Remark 3.2.1.</u> (i) If $\dfrac{f_{\underline{\lambda}_1}(x)}{f_{\underline{\lambda}_2}(x)}$ is nondecreasing in x for $\underline{\lambda}_1 \underset{m}{<} \underline{\lambda}_2$, then

(3.2.35) is satisfied, when (1) $F_{\underline{\lambda}}(x) = F(x-\xi(\underline{\lambda}))$, $-\infty < \xi(\underline{\lambda}) < \infty$

and $h(x) = x+d$, $d > 0$ where $\xi(\underline{\lambda})$ is Schur-concave in $\underline{\lambda}$, or when

(2) $F_{\underline{\lambda}}(x) = F(\frac{x}{\eta(\underline{\lambda})})$, $x \geq 0$, $\eta(\underline{\lambda}) > 0$, and $h(x) = cx$, $c \geq 1$ where

$\eta(\underline{\lambda})$ is Schur-concave in $\underline{\lambda}$.

(ii) If (3.2.35) is satisfied, then

$$(3.2.36) \qquad \sup\limits_{\Omega'} E_{\underline{\omega}}(S|R_h) = k \sup\limits_{\Omega_0} P[CS|R_h]$$

where $\Omega_0 = \{\underline{\omega} = (\underline{\lambda}_1,\ldots,\underline{\lambda}_k): \underline{\lambda}_1 = \ldots = \underline{\lambda}_k, \underline{\lambda}_i \in E, i=1,\ldots,k\}$

(iii) If (1) $F_{\underline{\lambda}}(x) = F(x-\xi(\underline{\lambda}))$, $-\infty < \xi(\underline{\lambda}) < \infty$ and $h(x) = x+d$

where $\xi(\underline{\lambda})$ is Schur-concave in $\underline{\lambda}$ or (2) $F_{\underline{\lambda}}(x) = F(\frac{x}{\eta(\underline{\lambda})})$, $x \geq 0$,

$\eta(\underline{\lambda}) > 0$ and $h(x) = cx$, $c \geq 1$ where $\eta(\underline{\lambda})$ is Schur-concave in $\underline{\lambda}$,

also if $\dfrac{f_{\underline{\lambda}_1}(x)}{f_{\underline{\lambda}_2}(x)}$ is nondecreasing in x for $\underline{\lambda}_1 \underset{m}{<} \underline{\lambda}_2$, then

$$(3.2.37) \qquad \sup\limits_{\Omega'} E_{\underline{\omega}}(S|R_h) = k P^\star.$$

3.3. A sufficient condition for the monotonicity in terms of weak majorization

First we give the definition of weak majorization.

<u>Definition 3.3.1.</u> A vector $\underline{a} = (a_1, \ldots, a_p)$ is said to weakly majorize a vector $\underline{b} = (b_1, \ldots, b_p)$, if $a_1 \geq \ldots \geq a_p$, $b_1 \geq \ldots \geq b_p$ and $\sum_{i=1}^{r} a_i \geq \sum_{i=1}^{r} b_i$, $r = 1, \ldots, p$. If $\underline{a}$ weakly majorize $\underline{b}$, then we write $\underline{a} \underset{m}{>>} \underline{b}$.

We state the following theorem which is the characterization of weak majorization (see Nevius, Proschan and Sethuraman [56]).

<u>Theorem 3.3.1.</u> $\underline{Z} \underset{m}{>>} \underline{Z}'$ if and only if $f(\underline{Z}) \geq f(\underline{Z}')$ for all nondecreasing Schur-convex functions, where $f(\underline{Z})$ is defined for $Z_1 \geq \ldots \geq Z_p$.

Let $\underline{X}_\lambda = (X_{\lambda_1}, \ldots, X_{\lambda_p})$ be a random vector of independent components where $\underline{\lambda} \subset E$ and the distribution function $F_{\lambda_i}(x)$ of random variable X_{λ_i} is stochastically increasing in λ_i, $i = 1, \ldots, p$.

Let

(3.3.1) $T = g(X_{\lambda_1}, \ldots, X_{\lambda_p})$ be a function of $X_{\lambda_1}, \ldots, X_{\lambda_p}$.

Let

(3.3.2) $F_\lambda(x) = P[T \leq x] = P[g(X_{\lambda_1}, \ldots, X_{\lambda_p}) \leq x]$.

In the following theorem, we obtain a sufficient condition for $E_\lambda(T, \underline{\lambda})$ to be nondecreasing and Schur-convex in $\underline{\lambda} \subset E$ for some function $\psi(T, \underline{\lambda})$ as defined in Theorem 3.3.2.

Theorem 3.3.2. If $\{F_{\underline{\lambda}}(x): \underline{\lambda} = (\lambda_1,\ldots,\lambda_p) \in E\}$ is a family of continuous distributions on the real line such that $F_{\underline{\lambda}}(x)$ is a differentiable function in x and each λ_i, $F_{\lambda_i}(x)$ is stochastically increasing in λ_i and $\psi(x,\underline{\lambda})$ is a bounded real-valued and differentiable function in x and λ_i, $i = 1,\ldots,p$, then $E\psi(T,\underline{\lambda}) = E\psi(g(\underline{X}_{\underline{\lambda}}),\underline{\lambda})$ is nondecreasing and Schur-convex in $\underline{\lambda} \in E$ provided

(3.3.3) (i) $\psi(g(\underline{X}_{\underline{\lambda}}),\underline{\lambda})$ is nondecreasing in $\underline{X}_{\underline{\lambda}}$ and $\underline{\lambda}$, and

  (ii) $\frac{\partial}{\partial x} F_{\underline{\lambda}}(x)(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})\psi(x,\underline{\lambda}) - \frac{\partial}{\partial x}\psi(x,\underline{\lambda})(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})F_{\underline{\lambda}}(x) \geq 0$,

    for $i < j$, $i,j = 1,\ldots,p$.

Proof. By Theorem 3.2.2.and (3.3.3)-(ii), $E\psi(T,\underline{\lambda})$ is Schur-convex in $\underline{\lambda}$.

   Now we want to show that $E\psi(T,\underline{\lambda})$ is nondecreasing in $\underline{\lambda}$. Let $\underline{\lambda} = (\lambda_1,\ldots,\lambda_p)$, $\lambda_1 \geq \cdots \geq \lambda_p$ and $\underline{\lambda}' = (\lambda_1',\ldots,\lambda_p')$, $\lambda_1' \geq \cdots \geq \lambda_p'$. Assume $\underline{\lambda} \geq \underline{\lambda}'$. We define $\underline{\lambda} \geq \underline{\lambda}'$ in the sense that $\lambda_i \geq \lambda_i'$, $i = 1,\ldots,p$. Since $F_{\lambda_i}(x)$ associated with the random variable X_{λ_i} is stochastically increasing in λ_i, i.e. if $\lambda_i \geq \lambda_i'$, $i = 1,\ldots,p$, then

(3.3.4) $X_{\lambda_i} \underset{st}{\geq} X_{\lambda_i'}$, $i = 1,\ldots,p$.

Since $X_{\lambda_1},\ldots,X_{\lambda_p}$ are independent random variables, then

(3.3.5) $\underline{X}_{\underline{\lambda}} \underset{st}{\geq} \underline{X}_{\underline{\lambda}'}$.

By (3.3.3)-(i), since $\psi(g(\underline{X}_{\underline{\lambda}}),\underline{\lambda})$ is nondecreasing in $\underline{X}_{\underline{\lambda}}$, then

(3.3.6) $\psi(g(\underline{X}_{\underline{\lambda}}),\underline{\lambda}) \underset{st}{\geq} \psi(g(\underline{X}_{\underline{\lambda}'}),\underline{\lambda})$.

Since $\psi(g(X_{\underline{\lambda}}\cdot),\underline{\lambda}')$ is nondecreasing in $\underline{\lambda}'$ and $\underline{\lambda} \geq \underline{\lambda}'$,

$$(3.3.7) \qquad \psi(g(X_{\underline{\lambda}}\cdot),\underline{\lambda}) \geq \psi(g(X_{\underline{\lambda}}\cdot),\underline{\lambda}').$$

From (3.3.6) and (3.3.7), we have

$$(3.3.8) \qquad \psi(g(X_{\underline{\lambda}}),\underline{\lambda}) \underset{st}{\geq} \psi(g(X_{\underline{\lambda}}\cdot),\underline{\lambda}').$$

Hence

$$(3.3.9) \qquad E\psi(T,\underline{\lambda}) \geq E\psi(T,\underline{\lambda}').$$

This completes the proof.

By applying Theorem 3.3.2, we can obtain the following results.

<u>Corollary 3.3.1</u>. If $F_{\underline{\lambda}}(x)$ is Schur-concave in $\underline{\lambda}$, $\psi(x,\underline{\lambda})$ satisfies (3.2.26), $\psi(x,\underline{\lambda})$ is nondecreasing in x, $\psi(g(X_{\underline{\lambda}}),\underline{\lambda})$ is nondecreasing in $X_{\underline{\lambda}}$ and $\underline{\lambda}$ and $F_{\lambda_i}(x)$ is stochastically increasing in λ_i, $i=1,\ldots,p$, then $E\psi(T,\underline{\lambda})$ is nondecreasing and Schur-convex in $\underline{\lambda}$.

<u>Corollary 3.3.2</u>. If $F_{\underline{\lambda}}(x)$ is Schur-concave in $\underline{\lambda}$, $\psi(x)$ is nondecreasing in x, $\psi(g(X_{\underline{\lambda}}))$ is nondecreasing in $X_{\underline{\lambda}}$ and $F_{\lambda_i}(x)$ is stochastically increasing in λ_i, $i=1,\ldots,p$, then $E\psi(T)$ is nondecreasing and Schur-convex in $\underline{\lambda}$.
By letting $\psi(x,\underline{\lambda}) = F_{\underline{\lambda}}^{k-1}(h(x))$ and using Theorem 3.3.1 and Theorem 3.3.2, we have the following theorem.

<u>Theorem 3.3.3</u>. Let $\{F_{\underline{\lambda}}(x), \underline{\lambda} \in E\}$ be a family of continuous distributions on the real line such that $F_{\underline{\lambda}}(x)$ and $F_{\underline{\lambda}}(h(x))$ are differentiable functions in x and λ_i and $F_{\lambda_i}(x)$ is stochastically increasing in λ_i, $i=1,\ldots,p$. Let $\psi(\underline{\lambda};c,d,k)$ be defined in (3.2.10).
Then $\psi(\underline{\lambda};c,d,k)$ is nondecreasing in $\underline{\lambda} \in E$ in the sense of weak majorization, i.e., it is nondecreasing and Schur-convex in $\underline{\lambda}$, provided that

(3.3.10) (i) $F_{\underline{\lambda}}(h(g(\underline{X}_{\underline{\lambda}})))$ is nondecreasing in $\underline{X}_{\underline{\lambda}}$ and $\underline{\lambda}$ and

(ii) $f_{\underline{\lambda}}(x)(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})F_{\underline{\lambda}}(h(x)) - h'(x)f_{\underline{\lambda}}(h(x))(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})$

$$F_{\underline{\lambda}}(x) \geq 0, \text{ for}$$

$i < j$, $i,j = 1,\ldots,p$, where $f_{\underline{\lambda}}(x) = \frac{d}{dx}F_{\underline{\lambda}}(x)$.

3.4. Selection of the population associated with $\underline{\lambda}_{[1]}$.

If the best population is defined to be the one associated with $\underline{\lambda}_{[1]}$, where $\underline{\lambda}_{[1]} \underset{m}{<} \underline{\lambda}_i$, $i = 1,\ldots,k$. We now define a class of procedures R_H for the selection of the population associated with $\underline{\lambda}_{[1]}$.

Let $H \equiv H_{c,d}$; $c \in [1,\infty)$, $d \in [0,\infty)$ be a function defined on the real line satisfying the following conditions. For every x

(3.4.1)　　(i) $H_{c,d}(x) \leq x$

　　　　　(ii) $H_{1,0}(x) = x$

　　　　　(iii) $H_{c,d}(x)$ is continuous in c and d

　　　　　(iv) $H_{c,d}(x) \uparrow \infty$ as $d \to \infty$ and/or $xH_{c,d}(x) \downarrow 0$ as $c \to \infty$.

Of particular interest are the functions $\frac{x}{c}$, $x-d$ and $\frac{x}{c} - d$. A class of procedures R_H for selecting a subset containing the best is defined as follows.

R_H: Select population π_i if and only if

(3.4.2)　　　　$H(T_i) \leq \min\limits_{1 \leq r \leq k} T_r$.

The probability of a correct selection is given by

(3.4.3) $P[CS|R_H] = \int_{-\infty}^{\infty} \prod_{r=2}^{k} \bar{F}_{\underline{\lambda}[r]}(H(x))dF_{\underline{\lambda}[1]}(x),$

where $\bar{F}_{\underline{\lambda}}(x) = 1-F_{\underline{\lambda}}(x).$

Because of the assumption (3.2.8),

(3.4.4) $P[CS|R_H] \geq \int_{-\infty}^{\infty} \bar{F}_{\underline{\lambda}[1]}^{k-1}(H(x))dF_{\underline{\lambda}[1]}(x).$

Let $\Omega = \{\underline{\omega} = (\underline{\lambda}_1,\ldots,\underline{\lambda}_k): \underline{\lambda}_i \in E, i=1,\ldots,k$ and there exists some

i such that $\underline{\lambda}_i \underset{m}{<} \underline{\lambda}_j \ \forall_j\}.$

Hence

(3.4.5) $\underset{\Omega}{\inf} P[CS|R_H] = \underset{\underline{\lambda}}{\inf} \varphi(\underline{\lambda};c,d,k)$

where

(3.4.6) $\varphi(\underline{\lambda};c,d,k) = \int \bar{F}_{\underline{\lambda}}^{k-1}(H(x))dF_{\underline{\lambda}}(x)$ and $\underline{\lambda} \in E.$

Using the same method of proof as in the case of R_h, we have the following results.

__Theorem 3.4.1.__ For the procedure R_H, $\varphi(\underline{\lambda};c,d,k)$ is Schur-convex in $\underline{\lambda} \in E$, provided that

(3.4.7) $(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})F_{\underline{\lambda}}(x)\cdot H'(x)f_{\underline{\lambda}}(H(x))-f_{\underline{\lambda}}(x)(\frac{\partial}{\partial\lambda_i} - \frac{\partial}{\partial\lambda_j})F_{\underline{\lambda}}(H(x))\geq 0,$
    for $i < j$, $i, j=1,\ldots,p$, where $H'(x) = \frac{d}{dx} H(x).$

Let $\Omega' = \{\underline{\omega} = (\underline{\lambda}_1,\ldots,\underline{\lambda}_k): \underline{\lambda}_i \in E, i=1,\ldots,k$ and there exist

$\ell_1,\ldots,\ell_k$ such that $\lambda_{\ell_1} \underset{m}{>} \lambda_{\ell_2} \underset{m}{>} \ldots \underset{m}{>} \lambda_{\ell_k}\}.$

__Theorem 3.4.2.__ $\underset{\Omega'}{\sup} E(S|R_H)$ takes place at $\underline{V}$ where $\underline{\lambda}_1 =\ldots= \lambda_k = \underline{V}$
provided that, for $\underline{\lambda}_1 \underset{m}{<} \underline{\lambda}_2$, $\underline{\lambda}_1,\underline{\lambda}_2 \in E,$

(3.4.8) $H'(x)f_{\underline{\lambda}_2}(H(x))(\frac{\partial}{\partial\lambda_{1i}} - \frac{\partial}{\partial\lambda_{1j}})F_{\underline{\lambda}_1}(x) - f_{\underline{\lambda}_2}(x)(\frac{\partial}{\partial\lambda_{1i}} - \frac{\partial}{\partial\lambda_{1j}})$

$$F_{\underline{\lambda}_1}(H(x)) \geq 0,$$

for $i < j$, i, $j = 1,\ldots,p$ where $\underline{\lambda}_i = (\lambda_{i1},\ldots,\underline{\lambda}_{ip})$, $i = 1,2$.

3.5. Selection procedures for multivariate normal distributions in terms of majorization.

Let $\pi_1,\ldots,\pi_k$ be k populations. Let π_i be associated with $\underline{X}_i' = (X_{i1},\ldots,X_{ip})$, $i = 1,\ldots,k$ where $\underline{X}_i$ is random vector with a p-variate normal distribution with unknown mean vectors $\underline{\mu}_i = (\mu_{i1},\ldots,\mu_{ip})$ and positive definite covariance matrix Σ. We assume that $\mu_{i1} \geq \ldots \geq \mu_{ip}$, $i = 1,\ldots,k$.

It is assumed that among the k given populations, there always exists $\underline{\mu}_{[k]}$ such that $\underline{\mu}_{[k]} \underset{m}{>} \underline{\mu}_{[i]}$ for all $i = 1,\ldots,k$. This population is called the best population.

Let $\underline{X}_{i1}',\ldots,\underline{X}_{in}'$ denote n independent observation vectors, each with p components, from population π_i.

Let

(3.5.1) $\underline{X}_{i\ell}' = (X_{i1\ell}, X_{i2\ell},\ldots,X_{ip\ell})$, $i = 1,\ldots,k$, $\ell = 1,\ldots,n$,

$$(3.5.2) \quad \bar{X}_{ij} = \frac{1}{n} \sum_{\ell=1}^{n} X_{ij\ell}, \quad i = 1,\ldots,k, \quad j = 1,\ldots,p,$$

$$(3.5.3) \quad Y_{ij} = \sum_{\ell=1}^{j} \bar{X}_{i\ell}, \quad i = 1,\ldots,k, \quad j = 1,\ldots,p,$$

$$(3.5.4) \quad S_{ij}^2 = \frac{1}{n} \sum_{m=1}^{n} (X_{ijm} - \bar{X}_{ij})^2, \quad i = 1,\ldots,k, \quad j = 1,\ldots,p,$$

$$(3.5.5) \quad R_{ij}^2 = \sum_{\ell=1}^{j} S_{i\ell}^2, \quad i = 1,\ldots,k, \quad j = 1,\ldots,p,$$

$$(3.5.6) \quad V_j^i = \mu_{i1} + \ldots + \mu_{ij}, \quad i = 1,\ldots,k, \quad j = 1,\ldots,p,$$

$$(3.5.7) \quad \underline{\mu}_{[i]} = (\mu_{(i)1},\ldots,\mu_{(i)p}), \quad i = 1,\ldots,k, \text{ and}$$

$$V_j^{(i)} = \mu_{(i)1} + \ldots + \mu_{(i)j}, \quad i = 1,\ldots,k, \quad j = 1,\ldots,p.$$

Let us denote by $\pi_{(i)}$ the population (unknown) associated with mean vector $\underline{\mu}_{[i]}$ and let $Y_{(i)j}$ be the observation associated with $\pi_{(i)}$, $i = 1,\ldots,k$, $j = 1,\ldots,p$. Our goal is to select a subset (may be empty) of the k populations so as to include the population associated with $\underline{\mu}_{[k]}$.

(A) Assume Σ is known.

We propose a selection rule as follows:

R_1: Select population π_i if and only if

$$(3.5.8) \quad Y_{ij} \geq \max_{1 \leq \nu \leq k} Y_{\nu j} - d, \quad j = 1,\ldots,p-1 \text{ and}$$

$$\min_{1 \leq \nu \leq k} Y_{\nu p} + d \geq Y_{ip} \geq \max_{1 \leq \nu \leq k} Y_{\nu p} - d.$$

Let

$$(3.5.9) \quad Q = D \Sigma^* D'$$

where

$$
\underset{((k-1)p)\times(kp)}{D} = \begin{pmatrix} A & 0 & \cdots\cdots\cdots & 0 & B \\ 0 & A & & \vdots & B \\ 0 & 0 & \ddots & \vdots & \vdots \\ \vdots & \vdots & & A & 0 & B \\ 0 & 0 & \cdots & 0 & A & B \end{pmatrix} ,
$$

$$
\underset{(kp)\times(kp)}{\Sigma^*} = \begin{pmatrix} T & 0 & & 0 \\ 0 & T & \ddots & \\ 0 & & \ddots & T \end{pmatrix} , \quad \underset{p\times p}{T} = \frac{1}{n} C\Sigma C'
$$

$$
\underset{p\times p}{A} = \begin{pmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{pmatrix}, \quad \underset{p\times p}{B} = \begin{pmatrix} -1 & & 0 \\ & \ddots & \\ 0 & & -1 \end{pmatrix} \text{ and } \underset{p\times p}{C} = \begin{pmatrix} 1 & & & 0 \\ 1 & 1 & & \\ \vdots & & \ddots & 1 \\ 1 & \cdots & 1 & 1 \end{pmatrix} .
$$

Let $\Omega = \{\underline{\omega} = (\underline{\mu}_1,\ldots,\underline{\mu}_k): \underline{\mu}_i \in E_1, \ i = 1,\ldots,k$ and there exists some i such that $\underline{\mu}_i \underset{m}{>} \underline{\mu}_j \ \forall j\}$, where

(3.5.10) $E_1 = \{\underline{\lambda} = (\lambda_1,\ldots,\lambda_p): \lambda_1 \geq \ldots \geq \lambda_p, \ -\infty < \lambda_i < \infty, i=1,\ldots,p\}$

The following theorem gives the infimum of the probability of a correct selection.

Theorem 3.5.1.

(3.5.11) $\inf_{\Omega} P[CS|R_1] = P[Z_i \leq d, |Z_j| \leq d$, for $i \in \{1,2,\ldots,(k-1)p\} -$

$\{p,2p,\ldots,(k-1)p\}$ and $j \in \{p,2p,\ldots,(k-1)p\}]$,

where $(Z_1,\ldots,Z_{(k-1)p})' \sim N(\ \underset{(k-1)p\times 1}{\underline{0}} \ , \ \underset{(k-1)p\times(k-1)p}{Q} \ ).$

Proof.

(3.5.12) $P[CS|R_1] = P\{Y_{(k)j} \geq Y_{(\nu)j}-d, |Y_{(k)p}-Y_{(\nu)p}| < d, j=1,\ldots,p-1,$

$$\nu=1,\ldots,k-1\}.$$

Note $Y_{ij} \sim N(v_j^i, \frac{1}{n} c_j'\Sigma c_j)$, $i = 1,\ldots,k$, $j=1,\ldots,p$, where

$c_j' = (\underbrace{1,\ldots,1}_{j \text{ times}},0,\ldots,0)$, $j = 1,\ldots,p$.

Since $\mu_{[k]} \underset{m}{>} \mu_{[i]}$, $i = 1,\ldots,k-1$,

(3.5.13) $\underset{\Omega}{\inf} P[CS|R_1] = P\{Y_{kj}^\star \geq Y_{\nu j}^\star -d, |Y_{kp}^\star -Y_{\nu p}^\star|<d, j=1,\ldots,p-1,$

$$\nu=1,\ldots,k-1\},$$

where

(3.5.14) $(Y_{\nu 1}^\star,\ldots,Y_{\nu p}^\star)' \sim N(\underline{0},T)$, $\nu = 1,\ldots,k$.

Let $\underline{Y}_\nu^\star = (Y_{\nu 1}^\star,\ldots,Y_{\nu p}^\star)'$, $\nu = 1,\ldots,k$.

Hence from (3.5.14),

(3.5.15) $\underline{Y} = (\underline{Y}_1^\star,\ldots,\underline{Y}_k^\star)' \sim N(\underset{kp\times 1}{\underline{0}}, \Sigma^\star)$.

Let $Z_{\nu i} = Y_{\nu i}^\star - Y_{ki}^\star$, $\nu = 1,\ldots,k-1$, $i = 1,\ldots,p$.

Let

$$Z_\nu^\star = (Z_{\nu 1},\ldots,Z_{\nu p})' \quad \nu = 1,\ldots,k-1.$$

Since

$$\underset{(k-1)p\times 1}{\underline{Z}} = (Z_1^\star,\ldots,Z_{k-1}^\star)' = D\underline{Y} \text{ and from (3.5.15), then } \underline{Z} \sim N(\underline{0},Q).$$

This completes the proof.

Theorem 3.5.2. If $Q = D\Sigma^\star D' = (q_{ij})$ is positive definite with

$q_{11} = \ldots = q_{(k-1)p,(k-1)p} = \sigma^2$ and $q_{ij} = \sigma^2\rho$ when $i \neq j$, σ and ρ are

known, then

(3.5.16) $\sup\limits_{\Omega} E(S|R_1) = KP^*$ provided that $\inf\limits_{\Omega} P[CS|R_1] = P^*$.

Proof.

$$(3.5.17) \quad E(S|R_1) = \sum_{i=1}^{k} P\{Y_{i\ell} \geq Y_{\nu\ell} - d, \; Y_{\nu p} - d \leq Y_{ip} \leq Y_{\nu p} + d,$$
$$\nu \neq i, \quad \nu = 1,\ldots,k, \; \ell = 1,\ldots,p-1\}$$

$$= \sum_{i=1}^{k} P\{Z_{i\nu\ell} \leq (V_{\ell}^{i} - V_{\ell}^{\nu}) + d, |Z_{i\nu p}| \leq d,$$
$$\nu \neq i, \quad \nu = 1,\ldots,k, \; \ell = 1,\ldots,p-1\},$$

where $Z_{i\nu\ell} = (Y_{\nu\ell} - Y_{i\ell}) - (V_{\ell}^{\nu} - V_{\ell}^{i})$, $\nu \neq i$, $\nu, i = 1,\ldots,k$, $\ell = 1,\ldots,p$.

Thus

$$(3.5.18) \quad \underset{(k-1)p \times 1}{\underline{Z}_i} = \{Z_{i\nu\ell}\} \sim N(\underline{0},Q).$$

Let

$$B_i = \{\underline{Z}_i' : Z_{i\nu\ell} \leq d, |Z_{i\nu p}| \leq d, \ell = 1,\ldots,p-1, \nu \neq i, \nu = 1,\ldots,k\}.$$

Let

$$\underset{1 \times (k-1)p}{\underline{T}_i} = (T_{i\nu\ell}) \text{ where } T_{i\nu\ell} = V_{\ell}^{i} - V_{\ell}^{\nu}, \; \nu \neq i, \nu, \; i = 1,\ldots,k, \ell = 1,\ldots,p.$$

Hence

$$(3.5.19) \quad E(S|R_1) = \sum_{i=1}^{k} P[B_i + \underline{T}_i].$$

Since the joint density of $\underline{Z}$ which is defined in (3.5.18) is

Schur-concave (see Marshall and Olkin [50]) and since $\underline{y} \in B_i$ and

$\underline{x} \underset{m}{<} \underline{y}$ implies $\underline{x} \in B_i$, then by Theorem 2.1 of Marshall and Olkin [50],

$P[B_i + \underline{T}_i]$ is Schur-concave function of $\underline{T}_i$, $i = 1,\ldots,k$.

Since $\underset{m}{T_i} > (a_i,\ldots,a_i)$, $i = 1,\ldots,k$ where

$$a_i = \frac{1}{(k-1)p} \sum_{\nu,\ell} T_{i\nu\ell} = \frac{1}{(k-1)p} \{kA_i - \sum_{j=1}^{k} A_j\} \text{ and}$$

$$A_i = V_1^i + \ldots + V_{p-1}^i, \text{ then from (3.5.19)}$$

$$(3.5.20) \quad E(S|R_1) \leq \sum_{i=1}^{k} P[B_i + (a_i,\ldots,a_i)]$$

$$\leq \sum_{i=1}^{k} P[B_i + (a_{[k]},\ldots,a_{[k]})]$$

where $a_{[k]}$ is the largest value among $\{a_1,\ldots,a_k\}$. That is,

$\sup\limits_{\Omega} E(S|R_1)$ is obtained when $a_1 = \ldots = a_k$. If $a_1 = \ldots = a_k$, then

$A_1 = \ldots = A_k$. Hence $a_1 = \ldots = a_k = 0$.

Thus

$$\sup E(S|R_1) = \sum_{i=1}^{k} P[B_i] = k \ P^* \text{ provided that } \inf_{\Omega} P[CS|R_1] = P^*.$$

This completes the proof.

(B) Let $\Sigma = \begin{pmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{pmatrix}$ where σ is known. Without loss of generality, we can assume $\sigma = 1$. We propose a subset selection rule as follows.

R_2: Select π_i if, and only if,

$$(3.5.21) \qquad Y_{ij} \geq \max_{1 \leq \nu \leq k} Y_{\nu j} - \frac{d\sqrt{2j}}{\sqrt{n}} \text{ for } j = 1,\ldots,p-1 \text{ and}$$

$$\max_{1\le p\le k} Y_{\nu p} - \frac{d\sqrt{2p}}{\sqrt{n}} \le Y_{ip} \le \min_{1\le\nu\le k} Y_{\nu p} + \frac{d\sqrt{2p}}{\sqrt{n}}.$$

<u>Theorem 3.5.3.</u>

(3.5.22) $\quad \inf_{\Omega} P[CS|R_2] = P[Z_{\nu i} \le d, |Z_{\nu p}| \le d \text{ for } i=1,\ldots,p-1]$
$$\nu=1,\ldots,k-1$$

where $Z_{\nu i}$ ($\nu=1,\ldots,k-1$, $i=1,\ldots,p$) are standard normal random
variables with

(3.5.23) $\quad Cov(Z_{\mu i},Z_{\nu j}) = \begin{cases} [\dfrac{\min(i,j)}{\max(i,j)}]^{\frac{1}{2}} & \text{for } \mu = \nu \\[2mm] \dfrac{\min(i,j)}{2\sqrt{ij}} & \text{for } \mu \neq \nu. \end{cases}$

Proof.

(3.5.24) $P[CS|R_2] = P\{Y_{(k)i} \ge Y_{(\nu)i} - \dfrac{d\sqrt{2i}}{\sqrt{n}},$

$$Y_{(\nu)p} - \frac{d\sqrt{2p}}{\sqrt{n}} \le Y_{(k)p} \le Y_{(\nu)p} + \frac{d\sqrt{2p}}{\sqrt{n}},$$

$$i=1,\ldots,p-1, \nu=1,\ldots,k-1\}.$$

Let

$$Z_{\nu i} = \frac{Y_{(\nu)i} - Y_{(k)i} - (V_i^{(\nu)} - V_i^{(k)})}{\sqrt{2i}/\sqrt{n}}, \quad \begin{array}{l} \nu = 1,\ldots,k-1, \\[1mm] i = 1,\ldots,p. \end{array}$$

Since $(Y_{(\nu)i} - Y_{(k)i}) \sim N(V_i^{(\nu)} - V_i^{(k)}, \dfrac{2i}{n})$,

then $Z_{\nu i} \sim N(0,1)$.

Hence

(3.5.25) $\quad P[CS|R_2] = P\{Z_{\nu i} \le d + \dfrac{V_i^{(k)} - V_i^{(\nu)}}{\sqrt{2i}/\sqrt{n}}, |Z_{\nu p}| \le d, i=1,\ldots,p-1,$

$$\nu = 1,\ldots,k-1\}.$$

Since $V_i^{(k)} \geq V_i^{(\nu)}$, $i = 1,\ldots,k-1$, then

$$\inf_{\Omega} P[CS|R_2] = P[Z_{\nu i} \leq d, |Z_{\nu p}| \leq d, \begin{array}{l} i=1,\ldots,p-1 \\ \nu=1,\ldots,k-1 \end{array}]$$

where $Z_{\nu i}$, $\nu = 1,\ldots,k-1$, $i=1,\ldots,p$ are $N(0,1)$ random variables

with covariances as defined in (3.5.23). This completes the proof.

If $k = 2$ and $p = 2$, we can show that

$$(3.5.26) \quad \inf_{\Omega} P[CS|R_2] = \int_{-\infty}^{\infty} \phi\left(\frac{d-(\frac{1}{2})^{1/4}x}{(1-\sqrt{\frac{1}{2}})^{\frac{1}{2}}}\right)\left[\phi\left(\frac{d-(\frac{1}{2})^{1/4}x}{(1-\sqrt{\frac{1}{2}})^{\frac{1}{2}}}\right)-\phi\left(\frac{-d-(\frac{1}{2})^{1/4}x}{(1-\sqrt{\frac{1}{2}})^{\frac{1}{2}}}\right)\right]d\phi(x)$$

Theorem 3.5.4.

$(3.5.27) \quad P[CS|R_2] \geq \max\{[\phi(d)-\phi(-d)]^{p(k-1)}, 1-(p+1)(k-1)\phi(-d)\}.$

Proof. Let $\{Z_{\nu i}\}$ be defined as in Theorem 3.5.3, then

$(3.5.28) \quad P[CS|R_2] \geq P[Z_{\nu i} \leq d, |Z_{\nu p}| \leq d, i=1,\ldots,p-1, \nu=1,\ldots,k-1].$

Note that

$$(3.5.29) \quad P[\bigcap_{i=1}^{m} A_i] \geq 1 - \sum_{i=1}^{m}\sum_{j=1}^{m} P[B_{ij}]$$

where $A_1,\ldots,A_m$ denotes a sequence of events, and A_i' the event
complementary to A_i, such that $A_i' = \bigcup_{j=1}^{n} B_{ij}$, $i=1,\ldots,m$.
Hence from (3.5.28) and (3.5.29),

$$(3.5.30) \quad P[CS|R_2] \geq 1 - \sum_{\nu,i} P[Z_{\nu i}>d] - \sum_{\nu=1}^{k-1} P[|Z_{\nu p}|>d]$$

$$= 1-(p-1)(k-1)\phi(-d)-(k-1)2\phi(-d)$$

$$= 1-(p+1)(k-1)\phi(-d).$$

By [68] and from (3.5.28)

(3.5.31) $\quad P[CS|R_2] \geq P[|Z_{\nu i}| \leq d, \ \nu=1,\ldots,k-1, i=1,\ldots,p]$

$$\geq \prod_{\nu,i} P[|Z_{\nu i}| \leq d] = [\phi(d)-\phi(-d)]^{p(k-1)}.$$

From (3.5.30) and (3.5.31), we prove the theorem.

For given $P^*(\frac{1}{k} < P^* < 1)$, a conservative value of d (d > 0) can be obtained by letting

$$\max\{[\phi(d)-\phi(-d)]^{p(k-1)}, \ 1-(p+1)(k-1)\phi(-d)\} = P^*.$$

Let $d_1 > 0$ be the value such that $[\phi(d)-\phi(-d)]^{p(k-1)} = P^*$ and let $d_2 > 0$ be the value such that $1-(p+1)(k-1)\phi(-d) = P^*$, then the minimum d(> 0) satisfying the basic requirement is given by $d = \min\{d_1,d_2\}$.

Theorem 3.5.5.

(3.5.32) $\quad E(S|R_2) \leq k[\phi(d)-\phi(-d)]$

Proof.

(3.5.33) $\quad E(S|R_2)= \sum_{j=1}^{k} P \begin{cases} Y_{ji} \geq Y_{\nu i} - \dfrac{d\sqrt{2i}}{\sqrt{n}} \\ \\ Y_{\nu p} - \dfrac{d\sqrt{2p}}{\sqrt{n}} \leq Y_{jp} \leq Y_{\nu p} + \dfrac{d\sqrt{2p}}{\sqrt{n}}, \\ \\ \qquad \text{for } i=1,\ldots,p-1, \nu \neq j, \nu=1,\ldots,k \end{cases}$

Let $Z_{\nu i} = \dfrac{Y_{\nu i}-Y_{ji}-(V_i^{\nu}-V_i^{j})}{\sqrt{2i}/\sqrt{n}}, \quad i = 1,\ldots,p, \nu \neq j, \nu=1,\ldots,k.$

Then

$$(3.5.34) \quad E(S|R_2) = \sum_{j=1}^{k} P\{Z_{\nu i} \le d + \frac{V_i^j - V_i^\nu}{\sqrt{2i}/\sqrt{n}}, |Z_{\nu p}| \le d,$$

$$i=1,\ldots,p-1, \nu \ne j, \nu = 1,\ldots,k\}$$

$$\le \sum_{j=1}^{k} p\{Z_{\nu i} \le d + \frac{V_i^j - V_i^\nu}{\sqrt{2i}/\sqrt{n}} \text{ for } \nu \ne j, \nu=1,\ldots,k\} = \Delta_i \text{ (say)},$$

$$i=1,\ldots,p-1.$$

Also,

$$E(S|R_2) \le \sum_{j=1}^{k} P[|Z_{\nu p}| \le d, \nu \ne j, \nu=1,2,\ldots,k] = \Delta \text{ (say)}.$$

Since $P[Z_{\nu i} \le d + \frac{V_i^j - V_i^\nu}{\sqrt{2i}/\sqrt{n}} \quad \nu \ne j, \nu=1,\ldots,k]$

$$\le \frac{1}{k-1} \sum_{\substack{\nu=1 \\ \nu \ne j}}^{k} P[Z_{\nu i} \le d + \frac{V_i^j - V_i^\nu}{\sqrt{2i}/\sqrt{n}}],$$

then

$$(3.5.35) \quad \Delta_i \le \frac{1}{k-1} \sum_{j=1}^{k} \sum_{\substack{\nu=1 \\ \nu \ne j}}^{k} \phi(d + \frac{V_i^j - V_i^\nu}{\sqrt{2i}/\sqrt{n}}) \text{ (since } Z_{\nu i} \sim N(0,1))$$

$$= \frac{1}{k-1} \sum_{j=1}^{k} \sum_{\substack{\nu=1 \\ \nu \ne j}}^{k} \phi(d + \frac{A_{[j]i} - A_{[\nu]i}}{\sqrt{2i}/\sqrt{n}})$$

$$= \frac{1}{k-1} \cdot Q \text{ (say)}$$

where $A_{[1]i} \le \cdots \le A_{[k]i}$ are order values of $\{V_i^1,\ldots,V_i^k\}$.
By the same argument in Gupta and Huang [34], we can show
that the sup of Q is obtained when $A_{[1]i} = \ldots = A_{[k]i}$.
Thus from (3.5.35)

$$(3.5.36) \quad \Delta_i \le \frac{1}{k-1} \sum_{j=1}^{k} \sum_{\substack{\nu=1 \\ \nu \ne j}}^{k} \phi(d) = k\phi(d), \quad i=1,\ldots,p-1.$$

Also

$$(3.5.37) \qquad \Delta \leq \sum_{j=1}^{k} \frac{1}{k-1} \sum_{\substack{\nu=1 \\ \nu \neq j}}^{k} P[|Z_{\nu p}| \leq d]$$

$$= k[\phi(d) - \phi(-d)].$$

Hence from (3.5.36) and (3.5.37),

$$(3.5.38) \qquad E(S|R_2) \leq \min\{k\phi(d),\ k[\phi(d) - \phi(-d)]\}$$

$$= k\{\phi(d) - \phi(-d)\}.$$

This completes the proof.

(C) Assume $\Sigma = \begin{pmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{pmatrix}$ where σ^2 is unknown.

We propose a subset selection rule as follows.

R_3: Select population π_i if and only if

$$(3.5.39) \qquad Y_{ij} \geq \max_{\nu} Y_{\nu j} - \frac{\sqrt{2}\, d\, R_{ij}}{\sqrt{n-1}}, \quad j = 1,\dots,p-1 \text{ and}$$

$$\max_{\nu} Y_{\nu p} - \frac{\sqrt{2}\, d\, R_{ip}}{\sqrt{n-1}} \leq Y_{ip} \leq \min_{\nu} Y_{\nu p} + \frac{\sqrt{2}\, d\, R_{ip}}{\sqrt{n-1}}.$$

Theorem 3.5.6.

$$(3.5.40) \qquad P[CS|R_3] \geq 1-(k-1)\sum_{i=1}^{p-1} p[t_i < -d] - (k-1)P[|t_p| > d]$$

where t_i is a r.v. having a t-distribution with $i(n-1)$d.f., $i=1,\dots,p$.

Proof.

$$(3.5.41) \qquad P[CS|R_3] = P[Y_{(k)j} \geq Y_{(\nu)j} - \frac{\sqrt{2}\, d\, R_{(k)j}}{\sqrt{n-1}},$$

$$Y_{(\nu)p} - \frac{\sqrt{2}\ d\ R_{(k)p}}{\sqrt{n-1}} \le Y_{(k)p} \le Y_{(\nu)p} + \frac{\sqrt{2}\ d\ R_{(k)p}}{\sqrt{n-1}},\ j=1,\ldots,p-1,$$

$$\nu=1,\ldots,k-1]$$

$$= P[t_{\nu j} \le d - \frac{V_j^{(k)} - V_j^{(\nu)}}{\sqrt{2}\ R_{(k)j}} \cdot \sqrt{n-1}, |t_{\nu p}| \le d, j=1,\ldots,p-1, \nu=1,\ldots,k-1]$$

where

$$(3.5.42) \quad t_{\nu j} = \frac{Y_{(\nu)j} - Y_{(k)j} - (V_j^{(\nu)} - V_j^{(k)})}{\sqrt{2}\ R_{(k)j}} \cdot \sqrt{n-1},\ j=1,\ldots,p, \nu=1,\ldots,k-1.$$

Since $(Y_{(\nu)j} - Y_{(k)j}) \sim N(V_j^{(\nu)} - V_j^{(k)}, \frac{2j}{n}\sigma^2)$

and $\frac{nR_{(k)j}^2}{\sigma^2}$ is distributed as a r.v. of χ^2 with $j(n-1)$ d.f., then $t_{\nu j}$

is a Student's t-distribution with $j(n-1)$ d.f.

Since $V_j^{(k)} \ge V_j^{(\nu)}, \nu = 1,\ldots,k-1,\ j = 1,\ldots,p$, then

$$(3.5.43) \quad \inf_{\Omega} P[CS|R_3] = P[t_{\nu j} \le d, |t_{\nu p}| \le d,\ j=1,\ldots,p-1, \nu=1,\ldots,k-1]$$

$$\ge 1 - \sum_{\nu,j} P[t_{\nu j} \le d] - \sum_{\nu=1}^{k-1} P[|t_{\nu p}| \le d] \quad \text{(by (3.5.29))}$$

$$= 1 - (k-1) \sum_{j=1}^{p-1} P[t_j \le d] - (k-1)P[|t_p| \le d]$$

where t_j is a r.v. of a t-distribution with $j(n-1)$ d.f., $j=1,\ldots,p$.
This completes the proof.

Theorem 3.5.7.

$$(3.5.44) \quad E(S|R_3) \le \min\{kG_1(d),\ k[G_p(d) - G_p(-d)]\}$$

where $G_j(x)$ is the c.d.f. of a r.v. of a t-distribution with $j(n-1)$

d.f., $j = 1,\ldots,p$.

Proof.

$$(3.4.45) \quad E(S|R_3) = \sum_{i=1}^{k} P \left\{ \begin{array}{l} Y_{ij} \geq Y_{\nu j} - \dfrac{d\sqrt{2}\, R_{ij}}{\sqrt{n-1}}, \\[3mm] Y_{\nu p} - \dfrac{d\sqrt{2}\, R_{ip}}{\sqrt{n-1}} \leq Y_{ip} \leq Y_{\nu p} + \dfrac{d\sqrt{2}\, R_{ip}}{\sqrt{n-1}}, \\[3mm] \text{for } j = 1,\ldots,p-1,\ \nu \neq i,\ \nu = 1,\ldots,k \}. \end{array} \right.$$

Let $t_{\nu j} = \dfrac{Y_{\nu j} - Y_{ij} - (V_j^{\nu} - V_j^{i})}{\sqrt{2}\, R_{ij}} \cdot \sqrt{n-1}$, then $t_{\nu j}$ is a Student's t-

distribution with $j(n-1)$ d.f.

$$(3.5.46) \quad E(S|R_3) = \sum_{i=1}^{k} P \left\{ \begin{array}{l} t_{\nu j} \leq d - \dfrac{V_j^{\nu} - V_j^{i}}{\sqrt{2}\, R_{ij}} \sqrt{n-1},\ |t_{\nu p}| \leq d, \\[3mm] j = 1,\ldots,p-1,\ \nu \neq i,\ \nu = 1,\ldots,k \end{array} \right\} .$$

Hence

$$(3.5.47) \quad E(S|R_3) \leq \sum_{i=1}^{k} P\{ t_{\nu j} \leq d + \dfrac{V_j^{i} - V_j^{\nu}}{\sqrt{2}\, R_{ij}} \cdot \sqrt{n-1},\ \nu \neq i,\ \nu = 1,\ldots,k \}$$

$$= \Delta_j \ \text{(say)},\ j = 1,\ldots,p-1.$$

and

$$(3.5.48) \quad E(S|R_3) \leq \sum_{i=1}^{k} P\{ |t_{\nu p}| \leq d,\ \nu \neq i,\ \nu = 1,\ldots,k \}$$

$$= \Delta \ \text{(say)}.$$

As in Theorem 3.5.5, we can show that the sup of Δ_j is obtained
when $V_j^1 = \ldots = V_j^k$, for $j = 1,\ldots,p-1$. From (3.5.47) and (3.5.48),
we get $\Delta_j \leq k G_j(d)$, $j=1,\ldots,p-1$ and $\Delta \leq k[G_p(d) - G_p(-d)]$. Thus

$$E(S|R_3) \leq \min\{ k[G_p(d) - G_p(-d)],\ k G_j(d),\ j=1,\ldots,p-1 \}$$

$$= \min\{ k[G_p(d) - G_p(-d)],\ k G_1(d) \}$$

3.6. Selection procedures for multivariate normal distributions in terms of weak majorization

It is assumed that among the k populations, there always exists $\mu_{[k]}$ such that $\mu_{[k]} \underset{m}{>\!\!>} \mu_{[i]}$, $i = 1,\ldots,k$. This population is called the best population. We are using the same notation as in Section 3.5. Let $\Omega = \{\omega \in (\mu_1,\ldots,\mu_k): \mu_i \in E_1, i=1,\ldots,k$ and there exists some j such that $\mu_j \underset{m}{>\!\!>} \mu_i \; \forall \; i\}$ where E_1 is defined in (3.5.10).

(A) Assume Σ is known.

We propose a selection rule as follows.

R_5: Select population π_i if and only if

(3.6.1.) $\qquad Y_{ij} \geq \underset{1 \leq \nu \leq k}{\max} \; Y_{\nu j} - d, \quad j = 1,\ldots,p.$

Using the same argument as in Theorem 3.5.1, we obtain the following result.

Theorem 3.6.1.

(3.6.2) $\quad \underset{\Omega}{\inf} \; P[CS|R_5] = P[Z_i \leq d, \; i = 1,\ldots,(k-1)p]$

where

$\quad (Z_1,\ldots,Z_{(k-1)p})' \sim N(\underset{(k-1)p \times 1}{0}, Q)$, where Q is defined in (3.5.9).

(B) Assume $\Sigma = \begin{pmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{pmatrix}$ where σ is known and we assume $\sigma = 1$.
We propose a selection rule as follows.

R_6: Select population π_i if and only if,

(3.6.3.) $Y_{ij} \geq \max_{1 \leq \nu \leq k} Y_{\nu j} - \frac{d\sqrt{2j}}{\sqrt{n}}$, j=1,...,p.

It is similar as in Theorem 3.5.3, we can show that

Theorem 3.6.2.

(3.6.4) $\inf_{\Omega} P[CS|R_6] = P[Z_{\nu j} \leq d, \nu=1,...,k-1, j=1,...,p]$

where $Z_{\nu j}$ is defined in Theorem 3.5.3.

If k = 2, p = 2, we can show that

(3.6.5) $\inf_{\Omega} P[CS|R_6] = \int \Phi^2 [\frac{d-(\frac{1}{2})^{1/4}x}{(1-\sqrt{\frac{1}{2}})^{\frac{1}{2}}}] d\Phi(x).$

In a same manner as in Theorem 3.5.4 and Theorem 3.5.5, we have the following results.

Theorem 3.6.3.

(3.6.6) $P[CS|R_6] \geq \max\{[\Phi(d)-\Phi(-d)]^{p(k-1)}, 1-p(k-1)[1-\Phi(d)]\}.$

Theorem 3.6.4.

(3.6.7) $E(S|R_6) \leq k\Phi(d).$

(C) Assume $\Sigma = (\begin{smallmatrix} \sigma^2 & & 0 \\ & \ddots & \\ 0 & & \sigma^2 \end{smallmatrix})$ where σ^2 is unknown.

We propose a subset selection rule as follows.

R_7: Select population π_i if and only if

(3.6.8) $Y_{ij} \geq \max_{\nu} Y_{\nu j} - \frac{\sqrt{2}\, d\, R_{ij}}{\sqrt{n-1}}$, j = 1,...,p.

In a manner similar to that in the proof of Theorem 3.5.6, we have

Theorem 3.6.5.

(3.6.9) $P[CS|R_7] \geq 1-(k-1) \sum_{i=1}^{p} P[t_i \leq -d]$

where t_i is the r.v. of Student's t-distribution with

$i(n-1)$ d.f., $i = 1,\ldots,p$.

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Chapter I of this thesis deals with some selection and ranking procedures for restricted families of probability distributions. A selection rule is proposed for distributions which are convex-ordered with respect to a specified distribution G. Some properties of this selection rule are derived. The asymptotic relative efficiencies of this rule with respect to other selection rules are evaluated. A selection rule is also proposed and studied for distributions which are s-ordered with respect to G. In Chapter II, some interval estimation→ next page

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problems for the unknown parameters of the k populations are studied. The infimum of the probability that a given confidence interval (based on suitably chosen order statistics) contains at least one "good" population is obtained. Different modifications and variations of this problem are also studied. The last section of Chapter II illustrates some results by means of two examples. Chapter III discusses the selection procedures in terms of majorization and weak majorization. The parameter is partially ordered by means of majorization or weak majorization. A class of procedures (R_h) for selecting the best population is defined. A sufficient condition is obtained for the infimum of the probability of a correct selection to be Schur-convex in λ. Also another sufficient condition for the same infimum of the probability of a correct selection to be nondecreasing and Schur-convex in λ is obtained. We also propose and study the selection procedures for multivariate normal distributions in terms of majorization and weak majorization. Various cases corresponding to the known or unknown common covariance matrix Σ are studied.

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✳ are discussed